Lecture 5
Solving Einstein’s equation
Some solutions (in brief)

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1 – Looking for solutions

The Einstein’s equation being highly non-linear, getting solutions is not an easy task. On the other hand, as for all the equations of physics, it can be solved using appropriate approximation methods.

→ in these conditions, **why interesting in exact solutions at all?**

The knowledge of exact solutions is of the **highest interest**, for different reasons:

- there is **no risk the solution to be spoilt by artefacts** resulting from some approximation scheme
- even if purely academic, an exact solution may **reveal some unexpected behaviour** that may turn out to be **specific to geometric gravity** (without any « equivalent » in Newton’s gravity, for instance)
- (last, but not least!) it turns out that **some of** the « simplest » of the **known exact solutions** (exhibiting a lot of symmetries: spherical, axial, homogeneous, …) are of **great relevance for astrophysics**: stars (internal structure & gravitational field), black holes, cosmology, …

How getting solutions?

Look for solutions exhibiting some **symmetries**!
→ the presence of symmetries:

- reduces the number of equations to solve
- allow to know partially the form of the metric (before solving the equations)
- …

… but … **take care! The symmetry does not always mean what you had first in mind!**

NOT THE SAME AT ALL !!!

One important ally: **gauge freedom**

- allows to choose a priori some of the metric functions, or some **extra equations** (ie having nothing to do with the Einstein's equation) these functions have to solve
- in a given gravitational field, the gauge equations **may be changed** in accordance with the question one has in mind!
- …

in this case, one ends up with more equations to solve (field + gauge eqs) → solving is simpler!

The **aim of this lecture**: just presenting some solutions of **astrophysical relevance**, with some of their main features (without any demonstration)

2 – The Schwarzschild's solution

The first (non trivial) known exact solution (1916). It is a **vacuum GR solution** (solves $R_{ab} = 0$)

Its most common form (Schwarzschild's coordinates) reads:

$$\underbrace{ds^2 = -\left(1-\frac{r_g}{r}\right)V^2 dt^2}_{\text{like Newton's metric with the 1/r (Newton's gravity) potential}} + \left(1-\frac{r_g}{r}\right)^{-1} dr^2 + \underbrace{r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)}_{\text{spacetime = a collection of 2 dimensional spheres (spherical symmetry)}} \quad \text{with} \quad \underbrace{r_g = \frac{2GM}{V^2} = 2m}$$

like Newton's metric
with the 1/r (Newton's gravity)
potential

spacetime = a collection
of 2 dimensional spheres
(spherical symmetry)

r_g = integration constant
r_g: rewritten as 2m for convenience
M = newtonian mass
M = m in relativistic units (G=V=1)

r_g=2m is refered to as the **Schwarzschild's horizon**

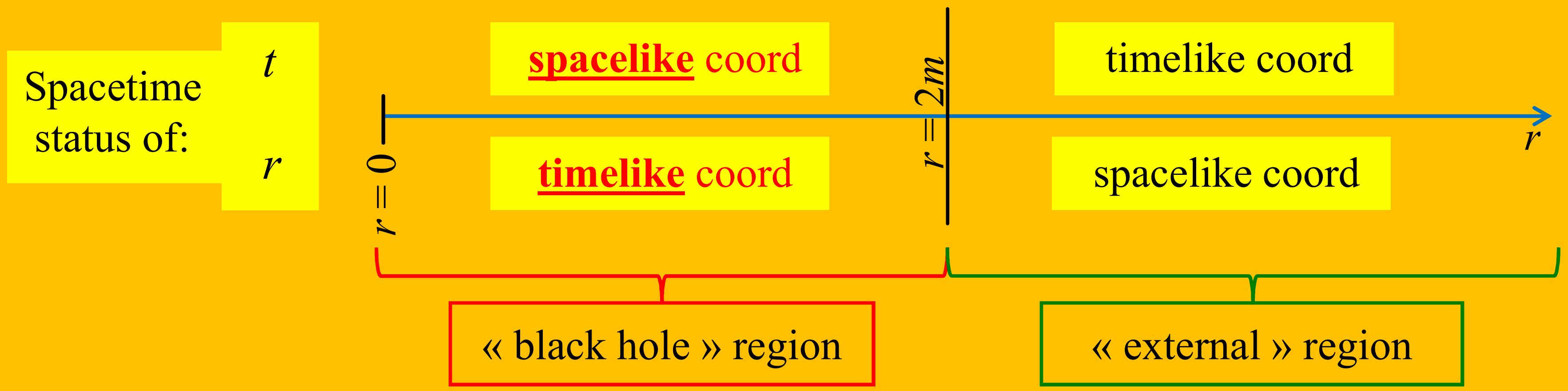
Let's remark that this solution:
- includes Minkowski as a special case (m=0)
- for all m, Schw is asymptotically Minkowski at large distances (r >> m)

Singularities:
- r = 0 is a **singularity** of the spacetime
- (maybe despite the apparences …) r = 2m is **NOT** a spacetime singularity!
  It is just a (say) coordinates' singularity

Other relevant (and sometimes a priori unsuspected!) properties of the Schw solution:

- for a given t, the surface of the points having the same coordinate r is $4\pi r^2$ (gauge freedom!)
- for $r_g > 0$ and $r \gg r_g$ the geodesics (free fall motion) can equivalently be interpreted as newtonian motions of test particles gravitationally attracted by an M massive body
- the Schw metric is the **unique** (GR) solution in the **vacuum** region surrounding a spherical star (Birkhoff theorem)
- the Schw metric is a solution whatever the sign of r_g (or m). But in the negative case, it can not be interpreted as being the external field of a star (at least if made of ordinary matter)

  → **one just considers the <u>$m > 0$</u> case in the following**
- the solution works both for $r >$ and $< r_g$
- for $r > r_g$ the metric is stationary
- for $r < r_g$ (if this spacetime region is vacuum, otherwise the Schw's metric is irrelevant), r is a timelike coord, while t is a spacelike one. This region is **<u>NOT</u> stationary** → **<u>black hole</u>** region!



Geodesics in Schw → free fall motion, ie motion of a test particle in the gravitational field of a spherical star (or of a non rotating black hole).

These equations can be written (planar motion, with A = 1-2m/r):

$$\frac{d^2t}{d\tau^2} + \frac{A'}{A}\frac{dt}{d\tau}\frac{dr}{d\tau} = 0 \quad \rightarrow \quad A\frac{dt}{d\tau} = E \quad \text{(cst)}$$

Energy conservation (of the test particle)

$$\frac{d^2r}{d\tau^2} + \frac{AA'}{2}\left(\frac{dt}{d\tau}\right)^2 - \frac{A'}{2A}\left(\frac{dr}{d\tau}\right)^2 - rA\left(\frac{d\varphi}{d\tau}\right)^2 = 0$$

$$\frac{d^2\varphi}{d\tau^2} + \frac{2}{r}\frac{dr}{d\tau}\frac{d\varphi}{d\tau} = 0 \quad \rightarrow \quad r^2\frac{d\varphi}{d\tau} = K \quad \text{(cst)}$$

Areal constant

$$\text{"light" case} \quad \rightarrow \quad d\tau = \sqrt{-ds^2}/V = 0 \quad \rightarrow \quad K = \infty$$

Relativistic Binet's equation: eliminate the **TWO** (« time ») quantities t & τ and get the differential equation whose solutions give the spatial description of the orbit (exact equation!):

$$\underbrace{\frac{d^2}{d\varphi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{m}{(K/V)^2}}_{\text{Binet/Newton equation (solutions = conics)}} + \underbrace{\frac{3m}{r^2}}_{\text{relativistic term/effects (solutions are no longer conics ...)}}$$

Binet/Newton equation
(solutions = conics)

relativistic term/effects
(solutions are no longer conics …)

the case of the "light"

$$\frac{d^2}{d\varphi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{3m}{r^2}$$

Let's compare some aspects of motions in a spherical gravitational field in Newton versus GR:

Newton's gravity	GR's gravity
- one can communicate with a « fay away observer » from any point of the spacetime	- one can communicate with a « fay away observer » **from points having r > 2m only** (ie from any point out of the « black hole region »)
- there are (just forced by gravity force) circular orbits for all r	- there are (inertial) **circular orbits for r > 3m** only (r=3m is the circular orbit for a Minkowski velocity particle)
- all the circular orbits are stable	- only **r > 6m circular orbits are stable**
- from any point, the escape velocity does not depend on the direction in which the particle is thrown	- the **escape velocity does depend on the direction** in which the particle is thrown (this dependence asymptotically vanishing far away)
- is light deflected by a gravitational field? The answer is not obvious, and depends on some extra hypotheses	- **the behaviour of light** (to the extent it moves at Minkowski's speed) **does not need any extra hypotheses** to be determined. Minkowski velocity particles follows null ($ds^2=0$) geodesics. Interpreting the far away asymptotical directions in newtonian terms (misleading!) would yield the conclusion that the « light is deflected »

Let's stress the historical importance of the Schw solution in the **first tests of GR**

Up to now, one just considered the spherical GR solution in vacuum. The solution does not work inside matter (the spherical star at the origin of the field, for instance, … if there is one!). Inside matter, the GR equation with stress tensor has to be solved. It is possible (spherical symmetry + gauge choice) requiring the metric to achive the form:

$$ds^2 = -A(t,r)V^2dt^2 + B(t,r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

the two metric functions A & B being got solving the field equation inside matter, that depend on the stress tensor that acts as the gravity source.

In the static case (A(r) & B(r) & no matter motion), B can be obtained and reads (using relativistic units):

$$\frac{1}{B(r)} = 1 + \frac{2}{r}\int_0^r T_0^0\, 4\pi r^2 dr$$

If all the matter is confined inside a sphere of radius R, the solution yieds, outside matter:

$$T_{ab}(r > R) = 0 \quad \rightarrow \quad \frac{1}{B(r > R)} = 1 + \frac{2}{r}\int_0^R T_0^0\, 4\pi r^2 dr \quad \xrightarrow{(sol\ =\ Schw)} \quad m = \int_0^R T_0^0\, 4\pi r^2 dr$$

In the case where the star is made of a perfect fluid (at rest), one gets: $m = \int_0^R \varepsilon\, 4\pi r^2 dr$

<u>**Two worthy remarks**</u>:

- while a static perfect fluid depends on two fields (energy density ε & pressure P), the pressure does not enter (in this very specific case) the expression of the mass
- the formula giving the mass rings a bell: it <u>**looks like**</u> the formula got in Newton's physics …

… but <u>***don't be confused!***</u> $4\pi r^2 dr$ is **NOT** the volume between the spheres r and r+dr !!!

(vacuum) Schwarzschild in other coordinates:

Isotropic coord: r' defined from the r Schw coord by: $r = \dfrac{m^2}{4r'}\left(1+\dfrac{2r'}{m}\right)^2$

$$ds^2 = -\left(\frac{1-\frac{m}{2r'}}{1+\frac{m}{2r'}}\right)^2 dt^2 + \left(1+\frac{m}{2r'}\right)^4 \times \underbrace{\left[dr'^2+r'^2\left(d\theta^2+\sin^2\theta d\varphi^2\right)\right]}_{=dx^2+dy^2+dz^2}$$

→ space part conformally euclidiean
→ does not penetrate inside the BH region (just a « part » of Schw)
→ isotropic coord are often used in weak field problems

Kruskal-Szekeres (Kruskal) coord:

defined from Schw coord by: $\begin{bmatrix} u \\ v \end{bmatrix} = \exp\left(\dfrac{r}{4m}\right)\sqrt{\left|\dfrac{r}{2m}-1\right|}\left\{\underbrace{\begin{bmatrix}\cosh \\ \sinh\end{bmatrix}}_{r>2m};\underbrace{\begin{bmatrix}\sinh \\ \cosh\end{bmatrix}}_{r<2m}\right\}\left(\dfrac{t}{4m}\right)$

$$ds^2 = \frac{32m^3}{r}\exp\left(-\frac{r}{2m}\right)\underbrace{\left(-dv^2+du^2\right)}_{\boxed{\text{2dim Minkowski}}}+r^2\left(d\theta^2+\sin^2\theta d\varphi^2\right)$$

with $r(u,v) \leftarrow \left(\dfrac{r}{2m}-1\right)\exp\left(\dfrac{r}{2m}\right)=u^2-v^2$

→ well suited for causal relations between events (← conformal Minkowski)
→ black hole nature/properties, gravitational collapse, …

3 – Some other exact solutions

The Schwarzschild-de Sitter solution:

Vacuum GR solution with Λ (solves $R_{ab} = \Lambda g_{ab}$)

$$ds^2 = -\left(1-\frac{2m}{r}-\frac{\Lambda r^2}{3}\right)V^2 dt^2 + \left(1-\frac{2m}{r}-\frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

- NOT asymptotically Minkowski
- **m=0 → de Sitter solution** (in Schwaezschild like coordinates)

Tolman-Oppenheimer-Volkoff (TOV) stars:

GR solution (without Λ) **inside** an incompressible (constant density ε) perfect fluid

$$ds^2 = -\left[\varepsilon + P(r)\right]^{-2} dt^2 + \left[1-\frac{8\pi}{3}\varepsilon r^2\right]^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

where P(r) is given by: $\dfrac{3P(r)+\varepsilon}{P(r)+\varepsilon} = \sqrt{\dfrac{3-8\pi\varepsilon r^2}{3-8\pi\varepsilon R^2}}$ where R = radius of the star

- a simple model for neutron stars (accepting ε = cst as a possible nucleus matter EOS)
- radius lower bounded by: R > 9m/4 (m = mass of the star)

The Kerr solution:

Vacuum RG solution with axial symmetry (no Λ)

- depends on two free parameters m (> 0) & a
  → referred to as the « mass » & «angular momentum per unit mass »
- $a = 0$ → Schw (Kerr = « Schw + rotation », in some sense …)
- describes (neutral) black holes with « angular momentum » (rotation)

It reads, in Boyer-Lindquist coordinates:

$$ds^2 = -dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2\right)\sin^2\theta d\varphi^2 + \frac{2mr}{\Sigma}\left(dt - a\sin^2\theta d\varphi\right)^2$$

with $\Sigma = r^2 + a^2\cos^2\theta$ & $\Delta = r^2 - 2mr + a^2$

Robertson-Walker solutions:

Metrics for (spatially) homogeneous & isotropic spacetimes: of first importance in **modern cosmology**!

Its general form reads, without any requirement on neither matter content nor gravity theory to be used:

$$ds^2 = -dt^2 + a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right] \quad \text{where} \quad k \in \{0, +1, -1\}$$

flat elliptic hyperbolic

If **GR framework** with no cosmological constant → Friedmann's models

The dust (P=0) euclidean (k=0) case (matter area): $a(t)=(6\pi m)^{1/3}(t-t_0)^{2/3}$ with $m=\varepsilon a^3=Cst$

The radiative area (P=ε/3) euclidean (k=0) case: $a(t)=2(2\pi E)^{1/4}(t-t_1)^{1/2}$ with $E=\varepsilon a^4=Cst$

Remote supernovae observations strongly suggest that our Universe:

- is spatially flat (k=0) …
- … and that its expansion is actually accelerated (→ Λ is not 0, in the usual GR description)

matter area solution → $a(t)=\left(\frac{8\pi Gm}{\Lambda}\right)^{1/3}\left[\sinh\left(\frac{\sqrt{3\Lambda}}{2}t\right)\right]^{2/3}$ with $m=\varepsilon a^3$

The late expansion is then exponential (de Sitter regime – in RW like coordinates –)

Kasner solution:

It describes a Universe that is homogeneous but **anisotropic**. It solves vacuum GR equation with Λ=0, and reads:

$$ds^2=-dt^2+t^{2p}dx^2+t^{2q}dy^2+t^{2r}dz^2$$

Three different expansion/contraction rates

with $\underbrace{p+q+r=p^2+q^2+r^2=1}$

Two « expansion » rates are > 0, one is < 0

4 – Linearized equations

In the case where a solution $\bar{g}_{ab}$ of a gravity theory is known, the method consists in looking for other solutions (of the same gravity theory) having the form: $g_{ab} = \bar{g}_{ab} + h_{ab}$ with h_{ab} "small" « small » meaning that the metric dependent quantities entering the field equations (curvature tensors, …) can be linearized in h_{ab}.

$\bar{g}_{ab}$ is refered to as the **background** solution, and h_{ab} as the **perturbation**

Gravitational waves (GW):

Consider the case:

- gravity theory: GR (without cosmological constant)
- background solution: Minkowski, as solution of the **vacuum** GR equation
- look for perturbed solutions of the **vacuum** GR equation

NOT necessarily! One could look for perturbed solutions inside matter as well! But then matter should result in a weak field in the considered spacetime region for perturbing about Minkowski to get sense (otherwise, perturb about something else …)

Let's use cartesian coordinates for the background metric, thence let's write (in relat units: V=1): $g_{ab} = m_{ab} + h_{ab}$ with $m_{ab} = \mathrm{diag}(-1,1,1,1)$ & $|h_{ab}| << 1$

One finds that, making relevant gauge choices, ie:

- first, Hilbert gauge, ie demand h_{ab} to satisfy the 4 extra conditions: $m^{ab}\partial_a\left(h_{bc}-\frac{1}{2}m_{bc}m^{de}h_{de}\right)=0$
- second, use the residual gauge freedom to specify to TT gauge (see later)

a solution reads:

$$ds^2=-dt^2+dx^2+dy^2+dz^2+h^+\left(dx^2-dy^2\right)+2h^\times dxdy \quad \text{where } h^+ \;\&\; h^\times = \text{any function of } (t-z)$$

This solution describes:

- a (weak) gravitational field that propagates along z at the unit, ie Minkowski, velocity
- that acts on particles in the directions that are **T**ransverse to the propagation axis z …
- … the perturbation being **T**raceless ($m^{ab}h_{ab}=0$)

→ **TT** gauge naming

Let's mention that the linearized (w.r.t. Minkowski) Einstein's equation reads, in vacuum, choosing the Hilbert gauge, reintroducing V (but not necessarily using TT coordinates):

$$\left(-\frac{1}{V^2}\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)h_{ab}=0$$

that shows that a gravitational perturbation propagates at the **Minkowski's speed** V in vacuum.
This is a very result of GR, which does not require extra hypotheses.
This is unlike the properties of electromagnetic waves, that require **choosing** an electromagnetic theory (including the way to adapt it in the GR framework). Choosing Maxwell theory (properly reformulated in the GR framework), one comes to the conclusion that:

Light propagates at the gravitational waves' speed (since at the Minkowski's speed …)

Other linearized schemes:

- background: Minkowski
- look for perturbed solutions inside matter

GW's generation by weak field sources theory

- background: Schwarzschild or Kerr
- look for vacuum perturbed solutions

Black holes' relaxation (towards equilibrium) theory

- background: Robertson-Walker
- look for perturbed solutions with the same matter content

Cosmological perturbations' theory

can be developped in both GR or alternative theories frameworks (possibly adaptating the background solution accordingly)

Beyond the linearized approach:

To be complete, let us just mention that it is possible going beyond a first order theory, writing the metric to be determined in an expanded form like: $g_{ab} = \overset{(0)}{g}_{ab} + \varepsilon \overset{(1)}{g}_{ab} + \varepsilon^2 \overset{(2)}{g}_{ab} + \ldots$

where ε is some « small parameter ». It can be $1/V$ (Post-Newtonian formalism), G (Post-Minkowskian formalism), … depending on the problem into consideration.

The other fields entering the problem generally have to be expanded in a similar way.