Lecture 4
General Relativity

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Table of contents

General relativity

1 – Lorentzian spacetimes

Coordinates

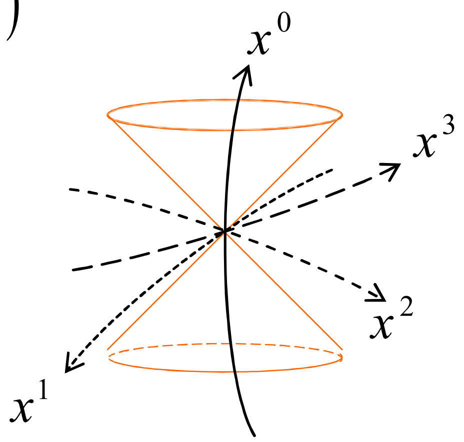
The spacetimes we consider are 4-dimensional, with coordinates written $\left(x^0, x^1, x^2, x^3\right)$

They are lorentzian spacetimes, with signature (-,+,+,+) (conventionally chosen to fit with the IAU recommandations, instead of the (+,-,-,-) often chosen in special relativity textbooks).

This means that at each event P, the metric is locally minkowskian. In other words, it is possible to choose a coordinates system such that the metric achieves, **at P and at P only** (a priori), the Minkowski form:

$$ds_P{}^2 = -V^2 dt^2 + dx^2 + dy^2 + dz^2 = -\left(dX^0\right)^2 + \left(dX^1\right)^2 + \left(dX^2\right)^2 + \left(dX^3\right)^2$$

Thence, **if the metric is diagonal** in a given coord system, one of the (diagonal) components of the metric tensor (say g_{00}) is negative, and the three other ones (g_{11}, g_{22}, g_{33}) are positive. This means that the curve on which only the x^0 coord varies lies inside the local Minkowski's cone, while the curves on which only one of the $\left(x^1, x^2, x^3\right)$ coord varies lie outside.



In this case, x^0 is said to be a temporal variable, since a displacement along it results in $ds^2 < 0$ (the definition of a time variable).
On the other hand, the three other variables (x^1, x^2, x^3) are said to be spatial variables, since a displacement along each of them results in $ds^2 > 0$ (the definition of a spatial variable).

However, **care should be taken** as these coordinates do not directly measure times intervals/distances that are measured by observers! See later …

The situation is even more delicate in the general case, where the metric is **not diagonal**. In this case, there is no constraint at all on the signs of the diagonal elements. In particular, the following situations may happen:

- 2 (or more) of the diagonal elements are negative → 2 (or more) coordinate curves inside the Minkowski's cone
- all the diagonal elements are positive → all the coordinate curves outside the Minkowski's cone
- some (or all) of the diagonal elements are = 0 → some (or all) of the coordinate curves tangent to the Minkowski's cone

… these circumstances having obviously no effect on the (-,+,+,+) signature of the spacetime (ie on its nature, this signature meaning « one time and 3 spatial directions »), thanks to the existence (in all of the above mentioned cases) of non diagonal terms.

Be that as it may, the metric determinant is a – 2 weighted density. Since the metric is locally minkowskian, and since the square of any jacobian is positive, det(g_{ab}) is then **negative** regardless to the used coord system. It is why the square root of the metric determinant, when needed in calculations, is often written $\sqrt{-g}$ instead of $\sqrt{|g|}$

In fact, from the **general theorem of geodesic frames**, it is even possible to get a bit more at any event P. Not only is that possible to ensure:

$$g_{00}(P) = -1 \;\; ; \;\; g_{11}(P) = g_{22}(P) = g_{33}(P) = 1 \;\; ; \;\; \text{the other} \quad g_{ab}(P) = 0$$

by an adequate coord choice, it is also possible to get in the same time:

$$\partial_c g_{ab}(P) = 0 \qquad (\Leftrightarrow \; \Gamma^c_{ab}(P) = 0 \;)$$

In such a coord system, **not only is the metric « Minkowski at P », but also is it only slowly going away from it** (quadratically) **when leaving P**.

Interpreting coordinates

Let's consider a first coord system $\left(x^0, x^1, x^2, x^3\right)$, and an event P. Let's also consider a second coord system (t,x,y,z) such that the metric achieves the Minkowski (in cartesian form) values at P (but not necessarily geodesic at P), and a particle that is instantaneously at rest in this locally minkowskian coord. The interval calculated along the orbit of this particle reads:

$$ds^2 = \underbrace{g_{ab}(P)dx^a dx^b}_{\text{as calculated in the first coord system}} = \underbrace{-V^2 dt^2}_{\text{as calculated in the second coord system}}$$

Thence, if t (minkowskian coord) is to be interpreted as the « **proper time** of the particle » under consideration (this point is **not obvious**, but is justified to the extent that the behaviour of its « internal clock » just couples with the metric), one concludes that the interval **ds²** between two close events can generally be interpreted as a measure of the (square of the) **proper time between** these **two events**.

However, the

geodesic motion, that should be interpreted as a **motion just determined by gravity**, since the only (geometric) properties of the spacetime determine it (inertial motion), of a particle crossing the event P is not rectilinear and uniform at P as measured by an observer at rest in the locally minkowskian coord system.

Indeed, separating its t and the (x,y,z) components, the geodesic equation reads, in these coordinates:

$$V\frac{d^2t}{d\lambda^2}+\gamma^0_{ab}(P)\frac{dX^a}{d\lambda}\frac{dX^b}{d\lambda}=0 \quad \& \quad \frac{d^2x}{d\lambda^2}+\gamma^1_{ab}(P)\frac{dX^a}{d\lambda}\frac{dX^b}{d\lambda}=0 \quad \& \quad \ldots \quad (\text{with } (X^a)=(Vt,x,\ldots))$$

(being the Christoffel at P calculated in the second coord system.) As just supposed, t measures the observer's proper time. Besides, thanks to the locally minkowskian character of the coordinates, the x,y,z coord can be interpreted as the ones giving the localisation and distances in terms of euclidean geometry (in the close neighbourhood of P, of course). Since the γ are not zero at P, the observer will conclude that, eliminating λ, the motion is **accelerated** at P in terms of its proper clock and measuring rods.

Let's now consider that the **second coord system** (t,x,y,z) **not only is minkowskian**, but that it is besides **geodesic at P**. In this case, considering a geodesic motion that crossing P, one now gets at P, from the geodesic equation:

$$V\frac{d^2t}{d\lambda^2}=0 \quad \& \quad \frac{d^2x}{d\lambda^2}=\frac{d^2y}{d\lambda^2}=\frac{d^2z}{d\lambda^2}=0$$

Thence, from the physical/geometrical interpretation of the locally minkowskian coordinates, an observer at rest in these coordinates comes to the conclusion that the particle is **not** accelerated. Knowing that the spacetime is not globally Minkowski, ie that both him and the particle are in a gravitational field, **he comes to the conclusion that he is itself in free fall in this gravitational field**.

Thence, in a geometric gravity theory, the existence of so called geodesic frames has this great physical interpretation:

it is always possible to locally (ie at any event prior fixed) cancel the effects of the gravitational field by an adequate choice of the coordinates system.

As a corollary, one gets the conclusion that the behaviour of a particle under a given gravitational field (thence, to the extent that the particle itself does not contribute to this gravitational field – test particle –) does not depend on the nature of this particle

→ **Free Fall Universality** (sometimes called « weak equivalence principle ») is back, as expected from a geometric gravity theory

Gauge freedom

There is **another** important point dealing with coordinates, that is not directly related to the previous discussion.
In Newton's/Minkowski's physics, the spacetime properties are known a priori. However, one can choose arbitrarily the coordinates to work with. This freedom is welcome and used, for instance, when problems exhibiting specific symmetries are into consideration: spherical spatial coordinates in the case of spherically symmetric sources, for instance.

This freedom may enter the game by different ways. Some of them are:

- choosing arbitrarily four functions (T,X,Y,Z) that define the new spacetime coordinates

$$t = T(t',x',y',z') \;\; ; \;\; x = X(t',x',y',z') \;\; ; \;\; y = Y(t',x',y',z') \;\; ; \;\; z = Z(t',x',y',z')$$

- let's remark that, instead of choosing explicitly (T,X,Y,Z), one could require them to satisfy 4 (more or less) arbitrary relations as well (under the form of algebraic relations, or partial differential equations, for instance)!
- choosing a priori the form of some of the (components of the) fields that otherwise should be got by solving the field equations. In some cases, that is indeed the true meaning of the choice of a peculiar coord system. For instance, when chosing spherical coordinates to determine the gravitational field of a spherical body in newtonian gravity: one in fact imposes the coord to be such that some of the components of this field vanish.

The essentially **new thing in geometric gravity** is that the spacetime properties (thence the metric tensor components) are not known a priori, and are then part of the functions to be determined (if not the only ones, in the case of vacuum solutions, for instance).

However, this fact does not prevent the possibility of imposing some prior conditions on the coordinates system (gauge freedom)**, under the form of relations constraining a priori some of the components of the metric tensor** (gauge constraints)

Since there are four coord, one has 4 degrees of freedom in the way of defining « new coord » in terms of « old » ones: $x^a\left(x'^b\right)$

But one can as well use this freedom to impose 4 conditions on the metric tensor components to be determined. The choice can be motivated by differents aims. For instance, in order to:

- simplify the calculations to do
- give some prior **geometrical** interpretations to some coordinates
- give some prior **physical** interpretations to some of the coordinates
- simplify the discussion of some worthy physical points
- …

It is worth to spot that, in a given spacetime, **the best suited coordinates system, or gauge constraints, depends on the problem one has to solve** in this spacetime. A gauge that is well suited to study the problem 1 may well prove to be not suited at all to solve the problem 2 (it is indeed often the case!)

Synchronous gauge

It consists in explicitly imposing a priori 4 components of the metric tensor: $g_{00} = -1$

$$g_{01} = g_{02} = g_{03} = 0$$

The metric then achieves the form: $$ds^2 = -dt^2 + g_{ij}\left(t, x^k\right) dx^i dx^j \quad \text{with} \quad i, j, k = 1,2,3$$

Advantages:

- t is a time coord everywhere, x^k are spatial coord everywhere
- not only is t a time coord, but it measures the proper time for any observer at rest (whatever its position)
- the time coord curves are geodesics

Obviously, the synchronous form is generally lost if a coord transform $x^a\left(x'^b\right)$ is made. However, it is easy to check that it is **NOT** the case for any transform of the form

time coord left unchanged & $x^{1,2,3}\left(x'^{1,2,3}\right)$

In fact, **the 4 gauge condition generally do not completely fix the gauge**: some residual freedom generally remain, ie some coord transforms remain possible that do not affect the gauge conditions (may be used to go further in the mathematical treatment of the problem).

These properties make the synchronous gauge well suited for many problem, in cosmology for instance.
On the other hand, in spite of these numerous advantages, this gauge is **not suited at all for the study of planetary motions!**

Let's also point out that a **gauge choice** is by no means a constraint on the geometrical properties of the spacetime under consideration, but **only a constraint on the way we represent this geometry** (ie on the coord used to explicit the ds^2).

Harmonic gauge

It consists in imposing 4 constraints on the components of the metric tensor, in the form of partial differential equations. It reads explicitly:

$$g^{ab}\Gamma^{c}_{ab} = 0 \qquad (\Leftrightarrow \partial_a\left(\sqrt{-g}\, g^{ac}\right) = 0 \ )$$

Its usefulness is not so obvious to give at this level. Let us just point out that its linearized version in weak field gravity makes obvious the propagative nature of gravitational perturbations (gravitational waves theory).
In this case too, the harmonic gauge condition does not completely fix the gauge.

2 – The stress tensor

In **Minkowski**'s spacetime, the stress tensor T_{ab} that described a physical system (perfect fluid, electromagnetic field, scalar field, …) is a second rank tensor that is built in such a way that:

- it is symmetric: $T_{ab} = T_{ba}$
- the impulsion-energy conservation of the physical system is recovered by requiring the stress tensor to be divergenceless, which reads, in cartesian coord: $\partial_a T^{ab} = 0$ (no care with the contracted index thanks to the symmetry)

Perfect fluids (in Minkowski)

The stress energy tensor of a perfect fluid reads: $T^{ab} = (\varepsilon + P) u^a u^b + P m^{ab}$

perfect fluid's … { … energy density (ε); … pressure (P); … four-velocity (u^a) }

m^{ab}: Minkowski's metric in cartesian coord

The associated conservation equations read:

$$(\varepsilon + P) u^\alpha \partial_\alpha u^\beta = -\left(m^{\alpha\beta} + u^\alpha u^\beta\right) \partial_\alpha P$$ Euler's equation

$$\partial_\alpha (\rho u^\alpha) = 0 \quad \text{where} \quad \rho = \exp\left\{\int \frac{d\varepsilon}{\varepsilon + P(\varepsilon)}\right\}$$ energy conservation

Let's stress (no pun!) that the possibility to recover the energy conservation requires an EOS (equation of state) besides the stress tensor, having the form $P = P(\varepsilon)$ (while recovering Euler's equation does not …)

Other examples

Electromagnetic (Maxwell's) stress tensor: $4\pi T^{ab} = F^{ac}F_c^{b} - \frac{1}{4} m^{ab} F_{cd}F^{cd}$ with $F_{ab} = \partial_a A_b - \partial_b A_a$

electromagnetic tensor

electromagnetic vector potential

Scalar field stress tensor: $8\pi T^{ab} = -m^{ab}\partial_a\phi\partial_b\phi - M^2\phi^2$

scalar field's mass

In spacetimes with curvature, the Minkowski's metric is generally just replaced by the spacetime metric g_{ab} and, accordingly, the partial derivatives by covariant ones, when needed (let's just mention that these extension are nevertheless neither unique nor obvious, other possibilities are indeed possible). In particular, the stress tensor « conservation » reads: $\nabla_a T^{ab} = 0$

In the case of a perfect fluid, for instance: $T^{ab} = (\varepsilon + P)u^a u^b + P g^{ab}$

… with the Euler's and energy conservations: $(\varepsilon + P)u^\alpha \nabla_\alpha u^\beta = -(g^{\alpha\beta} + u^\alpha u^\beta)\partial_\alpha P$

$$\nabla_\alpha (\rho u^\alpha) = 0 \quad (\Leftrightarrow \partial_\alpha (\sqrt{-g}\,\rho u^\alpha) = 0) \quad \text{where} \quad \rho = \exp\left\{\int \frac{d\varepsilon}{\varepsilon + P(\varepsilon)}\right\}$$

Let's remark that free particles may be described as a pressureless fluid (dust). In this case, Eulers's reduces to the **geodesics** equation. (One has $\rho = \varepsilon$ in the energy conservation equ.)

Let's also mention that the EOS for a ultra-relativistic fluids (asymptotically) reads: $P = \varepsilon/3$
… and that this EOS is (exactly) the one for a gas of photons

3 – The Einstein's equation

Newton's gravity theory proposes an equation that describes the way matter generates a field of force (in fact the potential the which it derives) inside Newton's spacetime, whose properties do not depend on its content. This equation reads: $\Delta U = -4\pi G\rho$
and is known as Poisson's equation.

Einstein's General Relativity (GR) **equation** describes a way matter, and generally its content, curves spacetime. More exactly, this equation says how the spacetime geometry is coupled to its physical content. This equation (as originally proposed) reads:

$$R_{ab} - \frac{1}{2} R g_{ab} = \chi T_{ab} \qquad \text{with} \qquad \chi = \frac{8\pi G}{V^4}$$

Einstein's tensor ($R_{ab} - \frac{1}{2} R g_{ab}$)

Einstein's gravitational constant (χ)

(global) stress tensor (T_{ab})

$\chi = \frac{8\pi G}{V^4}$: the link between Einstein's & Newton's gravitational constant for the induced Poisson's equation (in relevant conditions) to get its usual form

Some comments, remarks, …

In the case where the spacetime is filled with a single kind of « matter » (a perfect fluid, an electromagnetic field, …), T_{ab} is the stress tensor of this matter field. In the case where several such matter fields are present, T_{ab} is the sum of the stress energy tensor that describes each of these matter fields independently.

Thanks to Einstein's tensor zero divergence (← Bianchi identity), a **global** « conservation law » equation $\nabla_a T^{ab} = 0$ results as a direct consequence of the Einstein's equation.

In the case where the spacetime is filled by several **uncoupled** matter fields, the full theory (that would be got from a *lagrangian* formulation of GR) claims that each stress tensor is conserved:

$$R_{ab} - \frac{1}{2} R g_{ab} = \chi\left(\overset{(1)}{T}_{ab} + \overset{(2)}{T}_{ab} + \ldots \right) \quad \& \quad \nabla^a \overset{(1)}{T}_{ab} = 0 \quad \& \quad \nabla^a \overset{(2)}{T}_{ab} = 0 \quad \& \quad \ldots$$

These individual « conservation laws », when summed, return the global « conservation law » got from the Einstein's tensor zero divergence.

While Newton's gravity only depends on the matter density, Einstein's gravity depends on all the fields that characterize the matter content. In the case of a (perfect) fluid, not only the mass (energy) density does generate gravitation, but also pressure and the motion of the fluid (and other fields, in the case of non perfect fluids).

In Newton's theory, the laws of gravitation and of motion under gravity are independent laws. In Einstein's gravity, both are got in one shot: **Einstein's equation** in presence of a perfect fluid (for instance) yield at the **same level** the way the **spacetime geometry** couples with the fluid (gravitation law) and the **Euler's equation** that governs the motion of the fluid (law of motion).

In Newton's/Minkowski's physics:

- energy conservation ← uniformity of time
- impulsion conservation ← homogeneity of space
- angular momentum conservation ← isotropy of space

All these symmetries are (a priori) lost in the GR's spacetimes! Thence **there is no longer such conserved quantities** …

… unless some symmetry is present. For instance, in a stationary gravitational field, an energy conservation is recovered.

<u>The question of conserved physical quantities is by far not a trivial task in GR!</u> (out of the scope of these lectures.) The fact that the **divergencelessness of stress tensors** are customary referred to as « **conservation laws** » should not be misleading! Indeed, this naming is rather an inheritance of the fact that this divergencelessness means a true physical conservation law in **Minkowski's spacetime**, thanks to the fact that then only **partial derivatives** enter the corresponding equation. In GR spacetimes, the presence of **connection terms** in the **covariant derivatives** spoils the possibility to directly interpret these (local) equations in terms of some physical conserved quantities, once these equations are rewritten in integral form.

In Newtonian physics, the very notion of mass is a fundamental concept. Besides, the mass of an extended object is got by just summing the masses of its parts. **In GR, the situation is by far different**. A (say) static star made of (say) a perfect fluid is only defined by its distributions of energy density ε and pressure P. In this very special case, a global mass of the star can nevertheless be defined, as a quantity that is built up from the fields ε and P. But this « global mass » can not be (unambiguously) interpreted as being the sum of the masses of different parts of the star.

Considering more complex physical systems, **the situation is even more complex, since the very notion of the mass of the system can not be defined in a general way**. Neither can be notions like angular momentum, and such quantities. As fundamental concepts, these notions belong to the **newtonian framework of thinking**.

These « neonewtonian » notions are nevertheless **back in some particular situations** … that are, fortunately, **relevant in many astrophysical scenarios** …

Einstein's equation in inverse form

$$\boxed{R_{ab} - \frac{1}{2} R g_{ab} = \chi T_{ab}} \rightarrow \quad -R = \chi T \quad \rightarrow \boxed{R_{ab} = \chi \left(T_{ab} - \frac{1}{2} T g_{ab} \right)}$$

The two forms are equivalent. Using the second form, calculating the scalar curvature is not needed …

Vacuum GR equation: $\boxed{R_{ab} = 0}$

4 – The cosmological constant: historical versus modern sides

1915's successes of the just born GR where Newton's gravity failed:

- universality of free fall (Newton's theory requires an ad hoc hypothesis, while GR doesn't)
- does gravity need some delay to act on distant object?
  Newton: NO (sounds bad!); GR: YES (sounds nice!)
  (→ (GR's) gravitational waves theory)

conceptual problems
one doesn't like Newton's answers …
but **this does not prevent the theory** to work (coherence) and **to fit the observations!**

- **Mercury's orbital anomaly** →

by far a more « serious » problem, since
the possibility to fit the observations is at stake!

Current 1915's ideas on the « Universe as a whole » (cosmology!):

- today like it had ever been, and like it will be for ever: stationarity

fits well with Newton's ideas on spacetime …
… but what about the **history of matter inside it?**

- (spatially) finite/infinite?
  Olbers's paradox: why is the sky dark?
  → spatial finiteness as a way out?

do not fit with Newton's ideas on spacetime …
… but **does with GR's !!!**
→ worth trying to find a GR solution that would describe a **stationary & finite Universe!**

Required: stationarity + homogeneity + isotropy + matter = pressureless « gas of galaxies »

Minkowski does the job for stationarity + homogeneity + isotropy …
… but: spatially **infinite** & … Minkowski + GR eq → **vacuum**

→ how modifying $ds^2 = -V^2dt^2+dr^2+r^2(d\theta^2+\sin\theta^2 d\varphi^2)$ (Minkowski in spherical spatial coord) in order to get spatial sections having a finite volume?
Finite volume & homogeneity & isotropy → **3-sphere**

$$ds^2 = -dt^2 + \frac{dr^2}{1-\frac{r^2}{R^2}} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

(let's choose units such that $V = 1$, as usual in relativistic calculations …)

metric for a R radius 3-dim sphere

$$r = R\sin\psi$$
$$ds^2 = -V^2dt^2 + R^2\left[d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right]$$

That's for the spacetime geometry. What about matter?

dust → pressureless perfect fluid → $T^{ab} = \varepsilon u^a u^b \rightarrow T = g_{ab}T^{ab} = -\varepsilon$
(expected to modelize non interacting galaxies)

motion → at rest (in the coordinates just used to describe the expected geometry) → $(u^a) = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau}\right) = (1,0,0,0)$

since $d\tau^2 = -ds^2 = dt^2$

Calculate Christoffel's connection, then Ricci's curvature, and insert in Einstein's equation …

→ only 3 non trivial equations (4, but the 4rth is identical to the third) …

… one of them leading to **$\varepsilon = 0$**

No solution satisfying the requirements!

The way out? **Modify the field equation** in such a way that:

- the main properties of the field equations are preserved (second order PDE system, conservation laws, …)
- the modifications change essentially nothing at the solar system scale
- leads to a cosmological solution that satisfies the requirements

Solution (Einstein, 1917):

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = \chi T_{ab} \quad \leftrightarrow \quad R_{ab} - \Lambda g_{ab} = \chi \left(T_{ab} - \frac{1}{2} T g_{ab} \right) \qquad (\Lambda\text{GR equation})$$

cosmological constant (to be understood as a new fundamental constant of the nature)

divergenceless (thanks to (1) Bianchi's (2) Ricci's identities)

Induced Poisson like equation (makes sense, since the spacetime is lorentzian):

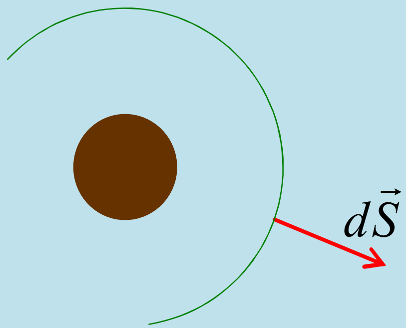
$$\Delta U = -4\pi G\rho + \Lambda$$

Consider a spherical mass (radius R) & integrate in a Gauss sphere of radius r (> R)

$$\iiint_{(Vol)} \vec{\partial}\left(\vec{\partial} U\right) dV = \oiint_{(Surf)} d\vec{S}.\vec{\partial} U = 4\pi r^2 \partial U$$

$$\iiint_{(Vol)} \rho\, dV = m$$

$$\iiint_{(Vol)} \Lambda\, dV = \frac{4}{3}\pi r^3 \Lambda$$



$$\partial U = -\frac{Gm}{r^2} + \frac{\Lambda}{3} r$$

(Seeliger-Newman equation)

Translate in newtonian terms:

global gravitational force has got 2 « components »:

(1) Newton’s usual $1/r^2$ force …

… that is attractive

(2) a new force, that is proportional to the distance …

… and **repulsive!** (if $\Lambda > 0$)

Two competitive forces (newtonian small scale interpretation)

→ existence of an « equilibrium distance », where the two components exactly compensate

A hope for a stationary cosmological model? The three Einstein's equations then yield (only two independent equations):

$$\Lambda = 4\pi G\varepsilon \qquad \& \qquad R = \frac{1}{\sqrt{\Lambda}} \qquad (\rightarrow \text{no solution if } \Lambda = 0\,!)$$

Thence the characteristics of the so called **Einstein's Universe** (metric and energy density):

$$ds^2 = -dt^2 + \frac{dr^2}{1-\Lambda r^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \qquad \& \qquad \varepsilon = \frac{\Lambda}{4\pi G}$$

Some years after, one discovered that **the Universe is expanding** (Slipher, Hubble, …)

- so sad for Albert E. …
- an elegant way out for « Olbers's paradox »!

Λ **no longer needed** …

… at least to get a stationary Universe!

The cosmological constant versus modern cosmology and modern physics

1998: one discovers the Universe's expansion is **accelerating**! (Perlmutter, Riess & Schmidt)

- acceleration from some billions years ago, after a first phase of decelerated expansion
- no possibility to explain this observed fact in the framework of:
  (1) usual (ie with $\Lambda = 0$) GR
  (2) ordinary (ie with positive pressure) matter
- on the other hand, this scenario is a **generic behaviour in the framework of GR with a (positive) cosmological constant** (and without having to call on negative pressure matter for help)

Let's remark that Λ can be formally interpreted as a particular negative pressure fluid! Indeed:

$$\left.\begin{array}{l}\text{Perfect fluid stress tensor: } T^{ab} = (\varepsilon + P)u^a u^b + P g^{ab} \\ \text{EOS: } \varepsilon + P = 0 \;\rightarrow\; P = -\varepsilon\end{array}\right\} \Rightarrow \quad T^{ab} = -\varepsilon g^{ab}$$

$$\nabla_b T_a^b = 0 \quad \text{with} \quad T_a^b = -\varepsilon \delta_a^b \quad \rightarrow \quad \partial_a \varepsilon = 0 \quad \rightarrow \quad \varepsilon = \text{constant}$$

Thence the ΛGR equation

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = \chi T_{ab}$$

can be **reinterpreted** as the GR equation without Λ, but with a matter content made of the usual matter described by the stress tensor T_{ab} (usual perfect fluid, or anything else) to which an extra perfect fluid like stress tensor, **with unusual EOS**, has to be added:

$$R_{ab} - \frac{1}{2} R g_{ab} = \chi T_{ab} - \Lambda g_{ab} = \chi \left(T_{ab} + \tilde{T}_{ab} \right) \text{ with } \tilde{T}_{ab} = \tilde{P} g_{ab} \quad \& \quad \tilde{P} = -\tilde{\varepsilon} = -\frac{\Lambda}{\chi} < 0$$

(the sign « < 0 » for a positive Λ). Note that one has not to wonder what the 4-velocity of this extra fluid should be, since $\tilde{\varepsilon} + \tilde{P} = 0$)

→ a fair interpretation of the cosmological constant as an exotic kind of « fluid »!

That's great!... But … just a **reinterpretation of ΛGR's equation** … unless some fundamental interpretation of such an exotic fluid would be found out!

Any proposition? YES!!! From the … QFT (Quantum Field Theory) community!

… which suggests us that **vacuum should be described** as some « potentiality of energy », that can be described **as a perfect fluid** with EOS that precisely reads:

$$\overset{(vac)}{P} = - \overset{(vac)}{\varepsilon}$$

… and some **experimental results** exist that support such a (seemingly crazy) claim!

If vacuum is actually some « potential energy », it should naturally enter the r.h.s. of the Einstein's equation. In its Λ = 0 version, but with ordinary matter T_{ab}, it should then read:

$$R_{ab} - \frac{1}{2} R g_{ab} = \chi \left(T_{ab} + \overset{(vac)}{T}_{ab} \right) \rightarrow R_{ab} - \frac{1}{2} R g_{ab} = \chi T_{ab} - \overset{(vac)}{\Lambda} g_{ab} \quad \text{with} \quad \overset{(vac)}{\Lambda} \equiv \chi \overset{(vac)}{\varepsilon} = cst$$

Along these lines, ΛGR is back, but with a **QFT interpretation of what Λ is**!...
… then with a proposition of what its **numerical value** should be …

not an obvious task:
divergent integral → cutoff → Λ^{QFT}

On the other hand, fitting the cosmological observations (cosmological redshifts of far away supernovae) with **ΛGR** cosmological models → Λ^{astro}

Up to that point, that is a very nice story. But things are going wrong when comparing the two values … Indeed, one gets:

$$\frac{\Lambda^{QFT}}{\Lambda^{astro}} \sim 10^{120} \quad !!!$$

Well, it may be just 10^{60} depending on the way the cut off is made, what contributes to vacuum energy … but it nevertheless remains the **HIGHEST DISCREPANCY ever observed in physics when trying to fit a theory with observational data** …

Ways out?

Because of this outstanding discrepancy, some people prefer trying **other solutions** rather than interpreting Λ as a manifestation of vacuum energy.

The 2 equivalent interpretations of ΛGR equation suggest **two main ways** to tackle the problem:

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta} \quad \leftrightarrow \quad R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 8\pi T_{\alpha\beta} + 8\pi \tilde{T}_{\alpha\beta} \left(\tilde{P} = -\tilde{\varepsilon}\right)$$

suggests :
don't change the Universe's matter content (r.h.s.), **but the gravity theory → alternative gravity theories**
reminiscent from
Mercury's perihelion's shift problem

- scalar-tensor gravity
- f(R) gravity
- …

suggests :
don't change the gravity theory (l.h.s.), **but the Universe's matter content → dark energy theories**
reminiscent from
Uranus' orbit's anomalies problem

- dark energy
- …

other options : change neither theory nor matter, but allow for

- voids (local inhomogeneities effect) ;
- remove large scale symmetries ;
- …