Lecture 2
Spacetime, inertia and gravitation

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1 – Preliminaries: Lagrangian & Lagrange's equations (variational calculous)

Consider:
- two fixed numbers *a* & *b;*
- a function *y*(*x*) that gets given values *A* & *B* for *x*=*a* & *x*=*b* respectively; [→ constraint ©]
- a functional, hereafter named lagrangian, that depends on the function y(x) and its first derivative.

Integrate this lagrangian between the two numbers *a* & *b*: $S[y]=\int_a^b L(y(x),y'(x))\,dx$

The result obviously depends on the choice of the function y(x).

→ Thence the question: among all the functions **such that *y*(*a*)=*A* & *y*(*b*)=*B***, how determining the ones for which *S* gets an (a local) extremum?

Let *y*(*x*) be a function satisfying ©. All the other function satisfying © can be written as *y*(*x*)+εη(*x*) where η(*x*) is a function such that η(*a*)= η(*b*)=0, and ε is a number that quantifies the deviation from *y*. The condition for *S* to experiment a (local) extremum reads

$$\forall\,\eta(x)\qquad S[y+\varepsilon\eta]-S[y]=\underbrace{o(\varepsilon)}\qquad (\text{ with }\quad \lim_{\varepsilon\to 0}[o(\varepsilon)]=0\ )$$

→ = 0 at the linear order in ε

$$S[y+\varepsilon\eta] = \int_a^b L(y+\varepsilon\eta, y'+\varepsilon\eta')\,dx = \int_a^b \left[L(y,y') + \varepsilon\eta\frac{\partial L}{\partial y} + \varepsilon\eta'\frac{\partial L}{\partial y'} \right] dx$$

$$S[y+\varepsilon\eta] - S[y] = \varepsilon\int_a^b \eta\frac{\partial L}{\partial y}\,dx + \underbrace{\varepsilon\int_a^b \eta'\frac{\partial L}{\partial y'}\,dx}_{\text{Integrate by parts}} = \varepsilon\int_a^b \eta\frac{\partial L}{\partial y}\,dx + \underbrace{\varepsilon\left[\eta\frac{\partial L}{\partial y'}\right]_a^b}_{=0 \text{ since } \eta(a)=\eta(b)=0} - \varepsilon\int_a^b \eta\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)dx$$

$$S[y+\varepsilon\eta] - S[y] = \varepsilon\int_a^b \eta\left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right]dx$$

… since the linear part in ε should be = 0 **whatever η**

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = \frac{\partial L}{\partial y}$$

Lagrange's equation

The Lagrange equation is a second order differential equation on *y* (since the lagrangian is just depending on the first derivative *y*').

Among the solutions, the one(s) that contains the points $(x,y) = (a,A)$ and (b,B) is (are) obtained by determining the corresponding values of the integration constants entering the general solution.

Two applications: (1) the shortest paths (geodesics) in the euclidean plane

In the euclidean plane, consider two points $(x,y) = (a,A)$ and (b,B) and the curves joining them.

Question: which one among these curves defines the shortest path (in term of usual euclidean distance)?

Representing these curves using cartesian axes, the distance reads

$$D = \int_{(a,A)}^{(b,B)} \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{1 + y'^2}\, dx = \int_a^b L(y, y')dx \qquad \text{with} \qquad L(y, y') = \sqrt{1 + y'^2}$$

The Lagrange equation then reads: $\dfrac{d}{dx}\left(\dfrac{\partial L}{\partial y'}\right) = 0 \;\rightarrow\; \dfrac{\partial L}{\partial y'} = cst \;\rightarrow\; y' = \alpha \;\rightarrow\; y = \alpha x + \beta$

→ The curves fulfilling the condition are then straight lines (… as expected, isn't it?).

The integration constants (α,β) are easily obtained from $\alpha a + \beta = A$ & $\alpha b + \beta = B$.

Two applications: (2) Newton's motion law in a potential

Consider a lagrangian depending on three independent functions of t (see the dedicated exercise for such a generalisation) having the form (the « prime » denoting the derivation w.r.t. t) :

$$L(x,y,z,x',y',z') = \frac{1}{2}m\left(x'^2+y'^2+z'^2\right)+\Phi(x,y,z)$$

The Lagrange equations read:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial x'}\right) = \frac{\partial L}{\partial x} \quad \rightarrow \quad mx'' = \partial_x \Phi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y'}\right) = \frac{\partial L}{\partial y} \quad \rightarrow \quad my'' = \partial_y \Phi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial z'}\right) = \frac{\partial L}{\partial z} \quad \rightarrow \quad mz'' = \partial_z \Phi$$

The Newton's law of motion of a particle of mass *m* moved by a potential Φ is back!

→ a (fruitful!) « variational reinterpretation » of Newton's dynamics

2 – Minkowski's spacetime: inertial orbits as a variational problem

Consider a timelike curve in Minkowski's spacetime. Along such a curve, the proper time between two (well separated) events on it reads

$$\tau = \frac{1}{V}\int_P^Q \sqrt{V^2 dt^2 - dx^2 - dy^2 - dz^2} = \int_{t_P}^{t_Q} dt\sqrt{1 - \frac{1}{V^2}\left(x'^2 + y'^2 + z'^2\right)}$$

It is easy to check that the corresponding Lagrange equations result in x, y and z affinely depending on the time t. Thence the result:

In the Minkowski's spacetime, inertial motions are the ones that extremise the proper time (that in fact render it maximal).

Let's point out that the result is the same as in Newton's spacetime: inertial motions, being not constrained by forces, can just be determined by the spacetime properties.

- In **Newton**'s spacetime, these properties are the euclidean character of space and the uniformity of time;
- In **Minkowski**'s, these properties are finally the same (for each observer), but are more nicely replaced (and summed up) by the properties of the **geometry** of the spacetime (which in turn is flat -see forthcoming lectures-).

Terminology: curves that locally extremize $\sqrt{|ds^2|}$ of a metric space/spacetime are usually refered to as **geodesics** of the space/spacetime

3 – Newton's spacetime & gravitation versus masses

Newton's gravity works in **two steps** (once the spacetime's properties have been defined)

← **gravity** just as a **special case** of what can generate **dynamics**

Consider a body.

If « **nothing** » **is acting** on it → **inertial motion**, that reflects spacetime properties

(→ in Newton's spacetime: rectilinear & uniform)

→ if the body does **not** experiment an **inertial motion**,
one **interprets** this by saying that something, **refered to as a force**,
drives it **away from** its inclination towards **inertial motion**

Step 0 (works in the **Newtonian framework** –and in Minkowski's as well …–):

if physics is to be described in a spacetime whose properties are **known beforehand**, we can define a **coordinates system** (t,x,y,z) having properties that are **known once and for all** (nothing to do with the bodies filling the spacetime, the forces acting on them, …)

→ in Newton's spacetime: **euclidean spatial coordinates**

Inertial motion $\leftrightarrow$ $\vec{v}$ is a constant vector

Step 1 (dynamics theory): one needs a law stipulating how $\vec{v}$ is changed by a given force (that may act on different bodies), ie how to calculate $\frac{d\vec{v}}{dt}$ (acceleration)

→ Since the acceleration is an object having 3 components, let's:

- describe the force by another 3-components object, written $\vec{f}$, supposed to be proportional to the acceleration (all being vectors in the Newtonian space)
- define a quantity m_{in}, attached to the body, named inertial mass, whose inverse stipulates how much this force generates an acceleration.

$$m_{in} \frac{d\vec{v}}{dt} = \vec{f}$$

It is important to spot the **high degree of generality of Newton's dynamics**: it aims to rule the motion of any body (once its inertial mass is known) **whatever the kind of force** acting on it!

→ General Relativity will be by far **less ambitious** than Newton's dynamics …
… just being a theory aimed to **govern motions under gravity** …

… and as soon as another kind of interaction is (also) to enter the game,

Newton's dynamics will be back!!!

(in a way adapted to the properties of spacetime in general relativity –curvature–, of course)

Step 2 (theory of the force of gravitation): one needs a law stipulating how to calculate the force in the case of gravitation

→ the necessity to define:

- a quantity m_g, attached to each body, named gravitational mass, that stipulates how much this body generates, and responds to, a gravitational field

Let's suppose that the same gravitational mass **does the work for both** …

- the **spacetime dependence** on the way two bodies, each with its own gravitational mass, **act gravitationally to each other**

Newton's law →

$$\vec{f}_{\text{body 1 on body 2}} = -G\, m_1^g m_2^g \frac{\vec{r}_2 - \vec{r}_1}{\left\|\vec{r}_2 - \vec{r}_1\right\|^3}$$

- time dependence: **instantaneous** ← get sense thanks to the **independence on observers of time measurements!**
- spatial dependence: attractive & $1/r^2$

By the way … **why on earth bothering with *G* ?**
→ not necessary at all for the theory to work ! But … **a way for the gravitational & inertial masses** to get the **same unit** (the **only reason**!)

Various remarks (1):

Experiments → the ratio m_g/m_{in} is **the same for all the bodies**

→ possibility to adjust G in such a way that this ratio is equal to one, ie **to get** $m_g = m_{in}$

In the lines of what experiments suggest, it seems natural to add an ansatz (extra axiom) to Newton's gravity: **the mass that generates gravity is nothing but the inertial mass**.

Some remarks:

- there is no logical inconsistency in doing this. The resulting theory is quite viable …
- … but it sounds like marrying cats and dogs in some way …
- … and, more deeply, it feels like **acting that** gravity has something to do with inertia

Remark that if one were less ambitious than Newton was, and just wished to:

- build a « dynamics under gravity theory » (the very aim of General Relativity, after all …)
- in the Newtonian spacetime framework (unlike General Relativity!)

it would be sufficient to build a theory that directly gives the way a body is accelerated in the presence of other bodies, **with no reference to any inertial mass concept**. The (only!) equation of the theory would be:

$$\frac{d^2\vec{r}_k}{dt^2} = -\sum_{i \neq k} M_i \frac{\vec{r}_k - \vec{r}_i}{\left\|\vec{r}_k - \vec{r}_i\right\|^3}$$

- no necessity for a constant G
- « masses » M to be united L^3T^{-2}

This theory is in accordance with observations (Newton's times, and up to the end of 19th century …). Indeed, this theory is equivalent to the original Newton's dynamics (if it were not for the restriction to gravity), providing the ansatz $m_g = m_{in}$ is accepted once for all.

Various remarks (2):

Because of time relativity, postulating an instantaneous interaction would not get sense in the framework of the **Minkowski's spacetime**

→ a Newton like gravitation theory in special relativity would necessarily require a time delay for a gravitational source to act on different locations of spacetime

→ demanding a gravitation theory to obey the **causality principle** would require gravity to act only inside the future Minkowski's cone attached to the source of the gravitational field

In Newton's approach, it is since both inertial and gravitational masses are attached to the bodies themselves that one is lead to expect that each body should be characterized by a specific mass ratio m_g/m_i. The fact that experiments do not fit with this expectation strongly suggests that, in our true spacetime, the way a body responds to **gravity does not originate in the body itself**.

→ the way out? Try a theory in which the response to gravity locates « **outside** » **the body**

What is external to the bodies? The **spacetime itself**.

→ could gravity « just » be a property of spacetime itself?
If it turns out to be the case: motions under gravity = inertial motions …

… but then spacetime could **NOT** be Newton's, nor Minkowski's …

→ if spacetime is to be nevertheless « locally Minkowskian » (lab experiments!), its geometry can not be flat, for **inertial motions not to be like Newton's ones**

Gravitation as an effect of a (non trivial) spacetime geometry?

4 – A variational problem & Newton's dynamics under gravity

We know how to formulate Newton's dynamics (in the potential case) in a variational way:

$$x^{i}{''} = \frac{\partial U}{\partial x^i} \quad \leftrightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial x^{i}{'}}\right) = \frac{\partial L}{\partial x^i} \quad \text{with} \quad L = \frac{1}{2} x^{i}{'} x^{i}{'} + U\left(x^k\right)$$

where U is the potential per unit mass (we use Einstein's convention in the « kinetic » term).

On the other hand, one knows that, in the Minkowski framework, extremizing the proper time (that is nothing but Minkowski's –flat– geometrical invariant: $d\tau^2 = - ds^2$) leads to inertial motions (in the Newtonian/Minkowskian meaning):

$$x^{i}{''} = 0 \quad \leftrightarrow \quad \frac{d}{dt}\left(\frac{\partial L_M}{\partial x^{i}{'}}\right) = \frac{\partial L_M}{\partial x^i} \quad \text{with} \quad L_M = \sqrt{1 - \frac{1}{V^2}\left(x'^2 + y'^2 + z'^2\right)}$$

Remark also that as long as slow motions (ie with $v << V$) are into consideration, the Minkowski's lagrangian can be rewritten, at the lowest order in v^2

$$L_M = 1 - \frac{1}{2V^2}\left(x'^2 + y'^2 + z'^2\right)$$

ie, up to a meaningless (since it disappears taking the partial derivatives) constant, the Newton's dynamics lagrangian in the no potential case. **THENCE THE QUESTION:**
how **modifying the Minkowski's lagrangian**, by introducing the potential entering Newton's dynamics, in such a way that its linearized version with respect to v^2 and U **returns the Newton's dynamics** lagrangian (maybe up to a constant factor and additional constant terms)?

Answer: it is easy to check that $L_M^{(U)} = \sqrt{1 - \frac{2U}{V^2} - \frac{1}{V^2}\left(x'^2 + y'^2 + z'^2\right)}$ does the job!

How to interpret this in « ds² terms »? Or, in « proper time terms »?

It is easy to check that this modified lagrangian is the one that would have been got from the following modified expressions of proper time, or ds²:

$$-V^2 d\tau^2 = ds_N^2 = -\left(1 - \frac{2U}{V^2}\right)V^2 dt^2 + dx^2 + dy^2 + dz^2$$

(refered to as « Newton's metric » in these lectures … thence the « N » index!)

If now this Newton's metric is to be interpreted as defining the geometrical properties of the spacetime, one could wonder if this spacetime could in fact just be Minkowski, but in non cartesian coordinates? The answer is NO, as soon as U explicitly depends on the position (we will check this further, let's just admit it for now).

Let us take a bit time to sum up what has been done here:

we have got the conclusion that a Newtonian dynamical problem, specifically

the motion of a particle determined by a potential *U*

can be interpreted as a geometrical problem as well, specifically

the determination of the geodesics of a spacetime that differs from Minkowski's by just multiplying the time-time metric tensor component by (1-2*U*/*V*²)

This nevertheless deserves some remarks in order to be careful to what this means … and also what this does not!

- In some sense, this is a geometrical **re**interpretation of what gravity is (to the extent that *U* is the gravitational interaction potential). Making a **switching from physics to geometry**!...

- … but this should **by no means** be interpreted as a kind of geometric theory of gravitation (what General Relativity will be). Indeed, this theory is fundamentally Newton's, since we started from the (dynamical) Newton's theory to built the Newton's metric. So far, **no way for getting ds_N^2 without having Newton's theory beforehand** is available …

- The **equivalence** between the dynamical and geometrical problems is **not exact**, since terms of order $O((v^2)^2,U^2,Uv^2)$ have been left apart …

- … meaning that if, **for some reason** (such a reason do not exist so far), the Newton's metric would have to be regarded as **exactly** representing the gravitational field of an isolated spherical body, the prediction on test particles' orbits would differ from the ones got from the usual dynamical Newton's theory if $O((v^2)^2,U^2,Uv^2)$ terms are taken into account in the calculations. The **$O((v^2)^2,U^2,Uv^2)$**, that would make the difference between the two theories, would be refered to as **Post-Newtonian terms** in the modern terminology

- In the same lines, if Newton's metric would have to be regarded as giving the true nature of the spacetime surrounding an isolated body, the proper time ruling the tips of a clock at rest would no longer be just given by the « time coordinate » t. Indeed, $d\tau^2 = \left(1 - 2U / V^2\right) dt^2$ shows that the proper time separating two successive tips also involves the location of the clock. With reasonable extra assumptions, it is not difficult to check that this results in a Doppler like effect, of **gravitational origin** (1960's Pound & Rebka experiment, see exercise)

- Formulating the « motion under gravity » problem in geodesics terms rules out the inertial mass problem (and even the need to define inertial mass!) from the start, the resulting orbits just invloving the spacetime structure, no matter what the orbiting particles are made of …

- Any gravity theory that would **directly give** the geometry of the spacetime (and aiming to interpret gravity this way!), **ie without any prior reference** to Newton's theory, has to lead back to Newton's metric in the circumstances where Newton's theory applies … otherwise it would just be ruled out on the ground of experiments/observations from the start!

- **Last but not least …** Let's point out the **uttermost importance** of the fact that the **spacetime** itself, and not only its spatial sections, **is endowed with a geometry** for the dynamic/geometric correspondance to work.
Riemann (the guy who invented riemannian geometries…) tried to make a geometric gravity theory (in the mid 19th century). But he just had Newton's spacetime as a starting point. Thence he just had space to curve ... A glance to the form of Newton's metric makes obvious why he could not succeed … relevant mathematics, but **unsuitable physical grounds**!

5 – Making a geometric gravity theory?

So far, ones has collected good reasons to believe that taking a look at riemannian geometries to get a new understanding of the very nature of gravity could be a worth thing to do …

However, such a geometric gravity theory, to get sense, should be formulated in a way that does not require any reference to Newton's theory.

As the Newton's theory structure is:

matter fields → gravitational potential

Poisson equation: $\Delta U = -4\pi G \rho$

from the matter fields, just involves the mass density

one would expect, in a similar way, a geometric gravity theory to get a structure like:

matter fields → spacetime geometry

maybe not limited to mass density …

from left to right only ?

defined by the metric tensor g_{ab}

The Newton's metric should be recovered as the « lowest order solution » as soon as the conditions where Newton's gravity has proved to be successful are achieved. Namely:

- slow motions → $v << V$ for both sources & test particles
- weak gravitational fields → $\left|\Delta v_{\text{due to gravity}}\right| << V$ for any objects
- quasi-stationary fields → $\left|\frac{\partial_t(\text{fields})}{\partial_{x,y,z}(\text{fields})}\right| << V$ (requires $v_{sources} << V$)

One should be aware that the identification of gravity to the metric properties of the spacetime renders the mere formulation of problems by far less obvious than it was in Newton's gravity. Indeed, a question like « what is the gravitational field caused by an object having the mass distribution *schtroumpf* »:

- is clear in Newton's gravity, *schtroumpf* defining the internal structure of the object the gravity field of which we want to know …
- … but the **very knowledge of this internal structure** means that we are able to say what the distances between different parts of the object are. However, these distances explicitly **refer to the geometrical properties of the spacetime**. The point is that in a geometric theory of gravity, these properties are precisely what is to be determined by solving the problem into consideration, ie by getting what the gravitational field is!

Thence, in a geometric gravity framework, **we are really in a mess!**
one needs the **matter distribution to get** the resulting **gravitational field**,
but, **in the same time**,
one needs the **gravitational field** to describe its source, ie the **mass distribution …**

In some sense, one could say that it is only once the solution of a problem is known that one is able to say which problem has been solved !!!
We know bags of (General Relativity) exacts solutions the meaning of which we don't have the foggiest idea!!!!!!

The way(s) out? But … **is there only way(s) out?**

there are at least two cases where the situation is not that dark …

- **weak gravitational fields**:
  (1) describe the matter distribution as if the spacetime were Minkowski
  (2) solve the field equation and get the metric tensor, that should be close to Minkowski (otherwise there would be something incoherent somewhere …)
  (3) use the previous solution to refine the matter distribution …
  (4) … the play goes on!
  → approximation methods. But: convergence problems? Apparition of effects that look like physics, but are in fact artefacts of the approximation method? …

- **highly symmetric solutions**: in this case, the spacetime properties are partly known. (geometric properties of some subsets of the spacetime are know.)

For instance, in the case of **spherical symmetry**, one knows beforehand that the whole spacetime is finally nothing but **a collection of spheres** of 2 dimensions. We then may **take advantage of** a simple form of **the 2-sphere metric**.

$$\text{metric of the 2-sphere} \rightarrow \; ds^2 = d\theta^2 + \sin^2\theta d\varphi^2$$

Thence the metric of a spherical four dimensional spacetime is expected to achieve the form

$$ds^2 = A(x,y)dx^2 + B(x,y)dy^2 + 2C(x,y)dxdy + D(x,y)\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

Two remarks:

- one is by no means compelled to use coordinates such that the metric achieves this form. It is just a possibility …
- further changes of (x,y) coordinates may reduce the number of metric functions without affecting this form: indeed, a transformation like x=f(x',y') and y=g(x',y') will just change the functions A, B, C and D
  → opens the possibility to choose a priori some of these functions (such choices may be physically significant → use this carefully!)

But whatever the choice, two metric functions remain to be determined to fully know the spacetime properties, thence the way the matter is distributed