Lecture 1

Space and time in physics

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Space and time in physics

1 – Space, time, spacetime, invariant interval

space (3 dim) = the set of all the possible places (→ lengths, angles)

time (1 dim) = the set of all the possible dates (→ time intervals)

spacetime (4 dim) = the set of all the possible events

an event(t,x,y,z) = a date(t) + a place(x,y,z)

the natural framework for the physicist :
here take place the events and the laws of physics (that impose some links between different « things that occur » at different events)

Both: - (interpretation of) precise measurements
- fiable theories (to make precise « predictions »)

require a preliminary precise description of the spacetime properties

???

a spacetime theory is required

Thence a (seemingly) natural procedure:

- do (first) a **theory** of spacetime …
- … and (in a second time) physical theories

… or ...
… both at the same time ?

A clear distinction should be made between:

- the **true spacetime** (ours!): its **properties are … what they are**, but
 (1) we don't know them a priori; nevertheless (2) we can approximately assess to them by **experiments/observations**

- **theoretical spacetimeS**: their **properties are what we decide** them to be, in such a way that
 (1) we know them perfectly, and (2) we can make arbitrarily precise **predictions** (as long as we can manage with the calculations to do!)

Compare and get a conclusion on the relevance of the chosen theoretical spacetime

➔ **Theoretical spacetimes**: how characterising their properties?

Let P & Q be two close events.

The spacetimes we will define will (each) be *locally* characterized by an interval I(P,Q) between these two events that does not depend on the observer (invariant interval)

Operational meaning : consider **identical** rulers & clocks

Same « behaviour » when remaining close each to others

Then consider 2 observers Obs & Obs', each being equipped with:

- one of these cloks
- some of these rulers (for building an Oxyz axes system)

Obs → P(t,x,y,z) & Q(t+dt,x+dx,…) $I_{Obs}(P,Q) = R(t,x,y,z,dt,dx,dy,dz)$

Obs' → P(t',x',y',z') & Q(t'+dt',x'+dx',…) $I_{Obs'}(P,Q) = R(t',x',y',z',dt',dx',dy',dz')$

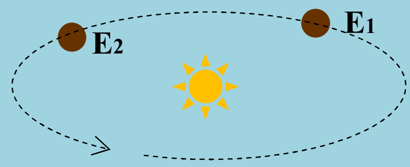
$$R(t,x,y,z,dt,dx,dy,dz) = R(t',x',y',z',dt',dx',dy',dz')$$

A remark:

Let's point out that an invariance like: $R(t, x, ..., dt, dx, ...) = R(t', x', ..., dt', dx', ...)$

generically results in:

- (a kind of) time relativity



- the existence of a velocity-like quantity that enters the very definition of the spacetime

… the precise form of which depending on (the chosen) R

… that does not have any prior physical meaning!
(so far, just a theory of spacetime, not of light or smth else!)

→ just a way to «combine» time & space like quantities !!!

Two examples of (theoretical) spacetimes: Newton's & Minkowski's

1a- Newton's spacetime : the invariant interval between two close events is the time intervalle between these two events

$$R(t, x, ..., dt, dx, ...) = dt \quad \rightarrow \quad dt' = dt$$

- absolute time
- simulaneity gets sense without any reference to the observer who makes measurements

➔ time & space measurements are disconnected

Newton's spacetime definition requires precisions on space besides the invariant interval

Newton (les « principia », 1687) :

- **time** is **uniform**
- **space** is **euclidean**

➔ **homogeneity** & **isotropy**

experimental meanings

local & global properties

remark that these properties **do not** logically **require** any link between dt and dt', like dt'=dt , or something else (ie a relation between time measurements made by different observers)

Remark: there is then no (non trivial) « time relativity » in Newton's spacetime.

← this (a priori rather exceptional) circumstance is due to the fact that the invariant does not involve spatial quantities (just dt …)

➔ while, **in Newton's spacetime**, the spatial separation between 2 events depends on the observer, **the time separation doesn't**

1b- Minkowski's spacetime : the invariant interval between two close events reads

$$ds^2 = -V^2 dt^2 + dx^2 + dy^2 + dz^2$$

new notation (instead of R)

V = Minkowski's speed (named this way in these lectures …)

It is noticeable that the Minkowski's spacetime properties are the same as the ones of Newon's (uniformity of time, euclidean space, both homogeneity & isotropy) apart from the fact that time (and space) measurements depend on the observer!!!

Unlike what happens in Newton's spacetime, the spatial section properties are given by the invariant interval. Indeed:

spatial sections of spacetime: $dt = 0 \;\rightarrow\; ds^2 = dx^2 + dy^2 + dz^2$

spatial sections are *locally* endowed with (3 dim) euclidean geometry

It is of first logical importance to spot (again) that *V* **does not have any prior physical meaning/interpretation**. So far, just a theory of spacetime is available, but neither a theory of light, electromagnetism, nor anything else.

However, let's point out **a remarkable property of the Minkowski's speed:**

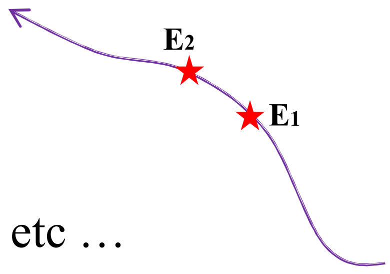
Consider any « objet » X that is observed by several observers O, O',O'', …, all of them being measuring the speed of X.

Generically, the measurements made by O, O',O'', … result in speeds that differ in both modulous and direction

BUT : it turns out that if one of the observers, say (O), gets the conclusion that the modulous of the speed is *V*, then

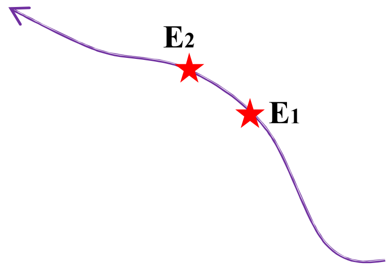
ALL THE OBSERVERS get the same conclusion: the SPEED MODULOUS IS *V* !!!

Proof: consider two (close) events E_1 & E_2 on the particle's orbit observed by two observers O and O'.



obs O: $E_1(t,x,\ldots)$ & $E_2(t+dt,x+dx,\ldots)$, ie E_1 occurs at time t on his clock, etc …
obs O': $E_1(t',x',\ldots)$ & $E_2(t'+dt',x'+dx',\ldots)$, ie …

The speed modulous measured by O being V, the invariant interval he calculates between the two events turns out to be zero. Indeed:



$$\left(ds^2\right)_{\text{obs O}} = -V^2 dt^2 + dx^2 + dy^2 + dz^2 = \left(-V^2 + \frac{dx^2 + dy^2 + dz^2}{dt^2} \right) dt^2 = 0$$

…the square of the speed… = V^2

Thence, from the invariance of this interval, the time and space intervals as measured by O' have to be such that

$$\left(ds^2\right)_{\text{obs O'}} = \left(ds^2\right)_{\text{obs O}} = 0$$

… that immediately leads to $\quad -V^2 dt'^2 + dx'^2 + dy'^2 + dz'^2 = 0 \quad \rightarrow \quad \frac{\sqrt{dx'^2 + dy'^2 + dz'^2}}{dt'} = V$

… meaning that O' also gets the conclusion that the (modulous of the) speed is V.

Remark1: this says **NOTHING** about which **physical « object »** V should be **the speed of**!

Remark2: only the **modulous** is concerned. Nothing ensures that dx'/dt' (resp dy'/dt', …) should be equal to dx/dt (resp dy/dt, …) …
… and generally it is not! → aberration phenomenon

A worth remark to have in mind, in relation with the following lectures …

The invariant interval: $ds^2 = -V^2 dt^2 + dx^2 + dy^2 + dz^2$

can be rewritten as well:

$$ds^2 = \sum_{a,b} g_{ab}\left(x^c\right) dx^a dx^b \qquad \text{notation} \qquad \left(x^0, x^1, x^2, x^3\right) \equiv \left(Vt, x, y, z\right)$$

$$= g_{ab}\, dx^a dx^b$$

omit the summation symbol on repeated indexes (Einstein's convention)

Minkowski

$$\left(g_{ab}\right) = \left(m_{ab}\right) \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$g_{ab}\left(x^c\right) =$ metric tensor (see later ...)

defines a (riemannian) **geometry**

Remark: one could rightly point out that the Newton's invariant may (also) be formally written as:

$$ds^2 = dt^2 = \left(\sum_{a,b}\right) g_{ab}\, dx^a dx^b$$

… but with:

Newton

$$(g_{ab}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The point is that **such a tensor g_{ab} does <u>NOT</u> define a riemannian geometry**
(← the corresponding matrix being not inversible)

Remark 1: a claim like « Newton's spacetime is flat » would then just be a **nonsense** …
(Newton's <u>space</u> is flat, but **Newton's spacetime has no geometry at all!!!**)

Remark 2: nevertheless, Newton's spacetime **<u>properties</u>** (and ability to make predictions) are, in some sense, ***as precise as*** the ones of Minkowski's …

The **<u>great lesson of special relativity</u>**, and one of the (if not THE) greatest discovery of all the physics:

The <u>spacetime</u> is endowed with a <u>geometry</u>!!!

2 – The relativity principle, Maxwell's theory & the Michelson-Morley experiment

The **relativity principle** (just a principle, not a logical necessity for physics !):

a **physical experiment** is unable to evidence an **absolute motion**

all kind of physical experiments ?
ie: are all the physical laws concerned ?

what is an « absolute motion »?

In « newtonian times » (until the end of 19th century):

an « absolute motion » is a motion with respect to a peculiar galilean frame, ie a way to distinguish a galilean frame among the other ones (galilean frames being each in rectilinear and uniform motion with respect to the others).

→ a physical law obeys the relativity principle if it is valid in all the galilean frames (Newton's or Minkowski's spacetimes framework)

The newtonian law of dynamics, and the Newton's gravitation theory, obey the relativity principle. These theories are formulated **in Newton's spacetime**.

→ **what about electromagnetism/optics?**

Maxwell's theory of electromagnetism

$$rot\,\vec{E} \equiv \vec{\partial} \wedge \vec{E} = -\partial_t \vec{B}$$
$$div\,\vec{B} \equiv \vec{\partial}.\vec{B} = 0$$

field properties not depending on the sources
→ E & B from potentials

$$div\,\vec{E} \equiv \vec{\partial}.\vec{E} = \rho / \varepsilon_0$$
$$rot\,\vec{B} \equiv \vec{\partial} \wedge \vec{B} = \mu_0 \left(\vec{j} + \varepsilon_0 \partial_t \vec{E}\right)$$

coupling fields & sources

→ vacuum solutions satisfy the propagation equation $\left(\Delta - \varepsilon_0 \mu_0 \partial_t \partial_t\right)\left(\vec{E}, \vec{B}\right) = 0$

meaning that the field propagates at (galilean) velocity of 3-modulus $c = \dfrac{1}{\sqrt{\varepsilon_0 \mu_0}}$

What does this mean? (1)

… this means that in any frame where the Maxwell's equations are valid (as they are written), the electromagnetic waves (Maxwell's interpretation of the nature of the light) travel space at a velocity that is fixed by the electromagnetic constants

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \cong 3.10^5 km / s$$

and that (consequently) their velocity does not depend on the direction of propagation
→ the propagation of light **is isotropic** (in any frame where Maxwell's equations are valid)

What does this mean? (2)

It is easy to check that, in the framework of the **Newton's spacetime**, if a set of objects fulfills these conditions in one galilean (say …) frame, it does no longer fulfill these conditions in any other frame that is in motion with respect to it.

→ if Maxwell's equations are valid in one galilean frame, they can not be in the others!

→ the Maxwell's theory **can not fulfill the relativity principle!** (in Newton's spacetime)

→ an electromagnetic expriment should make possible to **evidence an absolute motion!**

… then optic as well …

Thence, the electromagnetism is expected to defined a privilegied frame among the mechanically indistinguishable galilean frames. In this **privileged galilean frame**, and only in this one, is the Maxwell's theory (and the induced theory of optics) valid.

… named « aether »

Incidently, let's point out that only in this **aether** does the expression « **the** speed of the light » **make sense** …

The aim of Michelson-Morley experiment (1881):

→ **detecting the (yearly fluctuations of the) speed of the light as measured from Earth due to the Earth's motion around the Sun**

(by performing Michelson like interferometric observations of the light coming from a star during a whole year)

Michelson-Morley experimental result (1881):

➔ **no detection at all !!!**

Interpretation of this result (absence of detection):
whatever the motion of Earth, thence **whatever the** (instantaneous) **galilean frame** in which the experiment is performed, it appears that **the velocity of light IS isotropic!**

Such a conclusion is clearly incompatible with the Newton's spacetime framework …

On the other hand …

… the conclusion is compatible with the Minkowski's spacetime framework **iff the speed of light c is identified to the Minkowski's speed *V***

Thence, in the framework of Minkowski's spacetime, it is possible to understand the result of the Michelson-Morley's experiment, which strongly suggests that

$$c\left(=\frac{1}{\sqrt{\varepsilon_0\mu_0}}\right)=V$$

Accepting this, it turns out that
the **Maxwell's** electromagnetic (and optic) **theory obeys the relativity principle** …
… and it legitimates speaking of **THE speed of light !!!**

3 – More on Minkowski's spacetime …

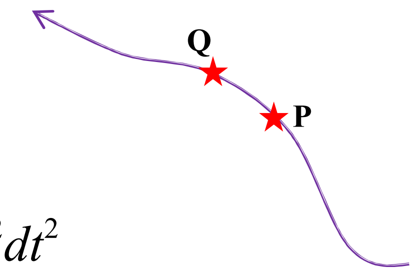
Once the choice that the electromagnetic constants and the Minkowski's speed are linked by

$$\varepsilon_0 \mu_0 V^2 = 1$$

has been made, it is legitimate to write c instead of V. However, we will keep the notation V in the framework of these lectures, when an explicit reference to this velocity is to be given.

<u>Minkowski's spacetime leads back to Newton's invariant for slow motions</u>

Whenever motions that occur at slow velocities (ie velocities <<V) are into consideration, the Minkowski's invariant reduces to Newton's in some sense. Indeed, considering two close events P & Q on a trajectory of a low velocity particle, the Minkowski's invariant reads, in terms of the particle's speed v



$$ds^2 = -V^2 dt^2 + dx^2 + dy^2 + dz^2 = -V^2 dt^2 + v^2 dt^2 = -\underbrace{\left(1 - \frac{v^2}{V^2}\right)}_{= 1 + O\left(\frac{v^2}{V^2}\right)} V^2 dt^2$$

which means that dt is invariant, up to second order (in v/V) terms.

Changing the coordinates

The invariant interval $ds^2 = -V^2dt^2+dx^2+\ldots$ characterizes Minkowski's spacetime, but also, in some sense, the spatio-temporal coordinates (t,x,y,z) used to localize events. However, an event may as well be represented by four other numbers, that are related to (t,x,y,z) by known relations, to be interpreted as defining a change of coordinates. Consider a general coordinates change:

$$t=T(t',x',y',z')\ ;\ x=X(t',x',y',z')\ ;\ y=Y(t',x',y',z')\ ;\ z=Z(t',x',y',z')$$

where T,X,Y and Z are 4 functions of 4 variables.

For instance

$$\begin{cases} t = t' \\ x = x'\sin y'\cos z' \\ y = x'\sin y'\sin z' \\ z = x'\cos y' \end{cases}$$

$$ds^2 = -V^2dt'^2+dx'^2 + x'^2\left(dy'^2+\sin^2 y'.dz'^2\right)$$

→ spherical (spatial) coordinates (no change in the time coordinate)

The invariant then takes the form

$$\begin{aligned} ds^2 &= -V^2\left(\frac{\partial T}{\partial t'}dt'+\frac{\partial T}{\partial x'}dx'+\ldots\right)^2+\left(\frac{\partial X}{\partial t'}dt'+\frac{\partial X}{\partial x'}dx'+\ldots\right)^2 \\ &\quad+\left(\frac{\partial Y}{\partial t'}dt'+\frac{\partial Y}{\partial x'}dx'+\ldots\right)^2+\left(\frac{\partial Z}{\partial t'}dt'+\frac{\partial Z}{\partial x'}dx'+\ldots\right)^2 \\ &=\left[-V^2\left(\frac{\partial T}{\partial t'}\right)^2+\left(\frac{\partial X}{\partial t'}\right)^2+\ldots\right]dt'^2+\left[-V^2\left(\frac{\partial T}{\partial x'}\right)^2+\left(\frac{\partial X}{\partial x'}\right)^2+\ldots\right]dx'^2+\ldots \\ &\quad+2\left[-V^2\frac{\partial T}{\partial t'}\frac{\partial T}{\partial x'}+\frac{\partial X}{\partial t'}\frac{\partial X}{\partial x'}+\ldots\right]dt'dx'+\ldots \end{aligned}$$

that, while a bit cumbersome, represents the Minkowski's spacetime as precisely and completely as the usual form $ds^2 = -V^2dt^2+dx^2+dy^2+dz^2$ does.

Question: among all these transformations, are there some which leave the ds^2 form unchanged, ie that are such that $ds^2 = -V^2dt'^2+dx'^2+dy'^2+dz'^2$?

Answer: yes there are … and these are named **Lorentz transformations**.

<u>The special lorentz transformations</u>: these are, among the general Lorentz transformations, the ones that leave the y & z coordinates « unchanged ». They write:

$$Vt' = \gamma\left(Vt - \frac{u}{V}x\right) \quad ; \quad x' = \gamma\left(x - \frac{u}{V}Vt\right) \quad ; \quad y' = y \quad ; \quad z' = z \quad \text{where} \quad \gamma = \left(1 - \frac{u^2}{V^2}\right)^{-1/2}$$

(t,x,y,z) defines a frame F, (t',x',y',z') a frame F'. In both, the invariant interval gets the « usual » form

$$ds^2 = -V^2dt^2 + dx^2 + dy^2 + dz^2 = -V^2dt'^2 + dx'^2 + dy'^2 + dz'^2$$

Terminology: the coordinates in which the invariant gets this form are called (pseudo)cartesian coordinates, and define (so called) galilean coordinate's systems.
t is refered to as the time coordinate of this galilean frame.
(x,y,z) are refered to as the spatial coordinates of this galilean frame.

Let's remark that if a particle is at rest in the frame F' (ie if x',y' & z' are constant), thence

$$dx' = \gamma\left(dx - \frac{u}{V}Vdt\right) = 0 \quad \rightarrow \quad \frac{dx}{dt} = u$$

in such a way that *u* can be interpreted as being « the velocity of F' with respect to F ». It turns out that considering Lorentz transformations gets sense only for $u < V$.

Some non cartesian forms of the Minkowski's metric

The change of coordinates (y & z unchanged)

$$t = \tau \cosh \psi \ ; \ x = V\tau \sinh \psi \ \rightarrow \ ds^2 = -V^2 d\tau^2 + V^2\tau^2 d\psi^2 + dy^2 + dz^2$$

… useful to describe some accelerated motions

Considering first Minkowski is spherical coordinates $ds^2 = -V^2dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta.d\varphi^2\right)$

the change of coordinates (angles unchanged)

$$Vt = \tau \cosh \chi \ ; \ r = \tau \sinh \chi \ \rightarrow \ ds^2 = -d\tau^2 + \tau^2\left[d\chi^2 + \sinh^2\chi\left(d\theta^2 + \sin^2\theta.d\varphi^2\right)\right]$$

… a « Robertson-Walker form » of Minkowski's spacetime

Sometimes refered to as the « Milne Universe » (akin to an expanding Universe metric)

The change of coordinates (y & z unchanged)

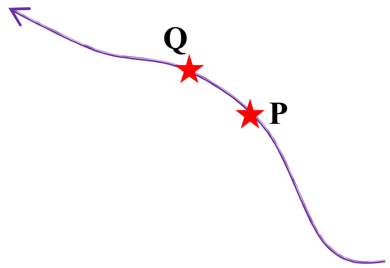
$$2Vt = u + v \ ; \ 2x = u - v \ \rightarrow \ ds^2 = -dudv + dy^2 + dz^2$$

… useful in some problems involving the propagation of light

CARE: the Minkowski's metric can not reach any prior given form. For instance **there is NO** coordinate change that would allow Minkowski to achieve the form

$$ds^2 = -d\tau^2 + \tau^2\left[d\chi^2 + \chi^2\left(d\theta^2 + \sin^2\theta.d\varphi^2\right)\right]$$

4 – Special relativity in (very) brief



Let's consider the motion of a « particle » in Minkowski.
In a given (say) cartesian coordinates system, it is customary to describe the corresponding orbit by the frame's time dependence of the frame's spatial coordinates

$$\left(x(t), y(t), z(t)\right)$$

and the derivatives of these functions define the (so named in these lectures) galilean velocity of the particle, customary written

$$\vec{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

The arrow on v could give to this set of three number the apparence of a vector (abusively, since a vector in the (4 dimensional) spacetime should have 4 components –see latter–).

Proper time

Consider two close events P & Q on the orbit. By definition the (Minkowski's) proper time between these two events is the time interval between these events as measured in a galilean frame F* in which the « particle » is (instantaneously) at rest.

$$ds^2 = -V^2 dt^2 + \underbrace{dx^2 + dy^2 + dz^2}_{= v^2 dt^2} = -\left(V^2 - v^2\right) dt^2$$

$$= -V^2 \left(dt^*\right)^2$$

at rest in F* → $dx^* = dy^* = dz^* = 0$

→ $dt^* = \sqrt{1 - \left(v / V\right)^2}\, dt$

Let's spot that, considering the motion of a « particle », the possibility to define a proper time attached to this particle necessitates that $ds^2 = -V^2 \left(dt^*\right)^2 \leq 0 \;\rightarrow\; v^2 \leq V^2$

and also, more precisely, that: $dt^* \neq 0 \;\rightarrow\; v^2 < V^2$

and also that, reciprocally: $v^2 = V^2 \;\rightarrow\; dt^* = 0$

that means that there is no proper time for a particle that moves at the Minkowski's velocity.

Note that, in terms of the Minkowski's metric tensor, the proper time also reads: $V dt^* = \sqrt{-ds^2} \;\rightarrow\; dt^* = \dfrac{1}{V}\sqrt{-m_{ab}\, dx^a dx^b}$

(4-)vectors displacement & velocity

In a coordinate system, the difference of the coordinates between two close events P & Q defines the displacement vector between these two events (note that it has 4 components). In cartesian coordinates, it simply reads (Vdt,dx,dy,dz). Note that it is customary to include V in the definition of the time component (while not logically necessary at all …).

The components of the so called 4-velocity vector (see latter for the justification of « vector » terminology) are the coordinates' variations per unit of **proper** time. In cartesian coordinates, it simply reads

$$\left(u^0,u^1,u^2,u^3\right)=\left(V\frac{dt}{dt^*},\frac{dx}{dt^*},\frac{dy}{dt^*},\frac{dz}{dt^*}\right)=\gamma\left(V,\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right)=\gamma\left(V,\vec{v}\right)\quad \text{with}\quad \gamma=\frac{1}{\sqrt{1-\left(v/V\right)^2}}$$

Remark that the norm (definition from the metric …) of the 4-velocity is (1) constant and (2) has the same value for all the possible motions

$$u^2\equiv m_{ab}u^a u^b = -\left(u^0\right)^2+\left(u^1\right)^2+\left(u^2\right)^2+\left(u^3\right)^2=\gamma^2\left(-V^2+v^2\right)=-V^2$$

Don't be confused by the misleading, but customary, notations …

Notation for the norm (does not prejudge for the sign !)

Notation for the component's index

Notation for the algebraic square (positivity !)

Digression: let's benefit the opportunity to « demystify » E = mc² …

Consider a particle whose (galilean) velocity changes (then so do the 4-velocity components). The constancy of the 4-velocity norm results in an appealing relation. Indeed, it leads to:

$$\gamma^2 V^2 - \gamma^2 \vec{v}^2 = V^2 \quad \rightarrow \quad 2\gamma V^2 d\gamma - 2\gamma\vec{v}\, d(\gamma\vec{v}) = 0 \quad \rightarrow \quad d(\gamma V^2) = \vec{v}\, d(\gamma\vec{v}) = d\vec{r}\, \frac{d}{dt}(\gamma\vec{v})$$

Attaching to the particle that describes the orbit a mass (dynamical concept, not to be developped in these lectures … we just need here to know that it is a number that characterizes the particle and does not depend on its motion), this relation writes:

$$\underbrace{d(\gamma m V^2)}_{\text{… the energy of the particle}} = d\vec{r}\,.\,\frac{d}{dt}\underbrace{(\gamma m \vec{v})}_{\text{… the impulsion of the particle}}$$

In relativistic dynamics:

to be identified with …

… the **energy** of the particle

… the **impulsion** of the particle

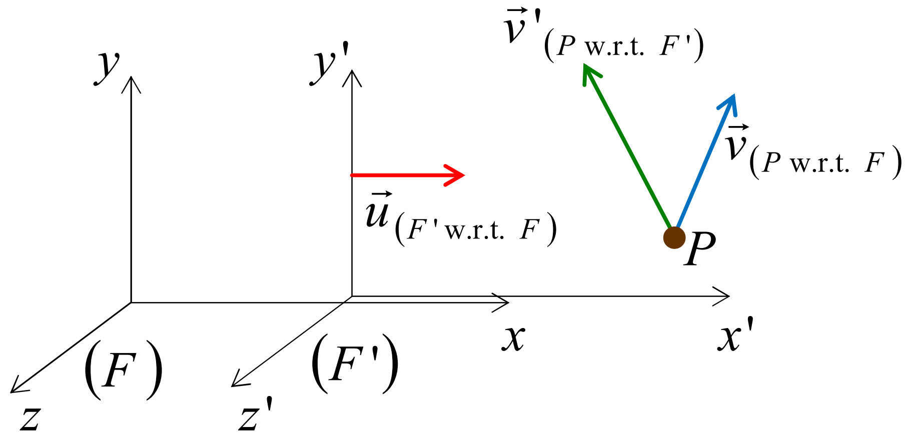
… the **force** acting on the particle

… this is then the **work of the force** in the elementary displacement

$$E = \gamma m V^2$$

In relativistic dynamics, $(\gamma m V, \gamma m\vec{v}) = \left(\frac{E}{V}, \gamma m\vec{v}\right) = m(u^0, u^1, u^2, u^3)$ is refered to as the **4-vector energy-impulsion** (it is a 4-vector as the 4-velocity is) of the (massive) particle. Its modulous $(m^2 V^2)$ returns the mass (or rest energy) of the particle.

Velocity composition



Consider two galilean frames F & F' related by a special Lorentz transform.
Consider a particle P moving with velocities:

- $\vec{v}$ w.r.t. F
- $\vec{v}\,'$ w.r.t. F'

How getting $\vec{v}\,'(v'_{x'}, v'_{y'}, v'_{z'})$ in terms of $\vec{v}(v_x, v_y, v_z)$ & u ?

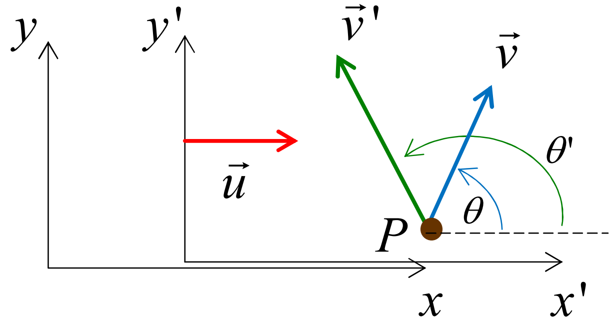
Velocity components measured in F: $v_x = \dfrac{dx}{dt}$, $v_y = \dfrac{dy}{dt}$, $v_z = \dfrac{dz}{dt}$

Velocity components measured in F': $v'_{x'} = \dfrac{dx'}{dt'}$, $v'_{y'} = \dfrac{dy'}{dt'}$, $v'_{z'} = \dfrac{dz'}{dt'}$

From the Lorentz special transform, one has, between two close events on P's orbit:

$$dt' = \gamma\left(dt - \frac{u}{V^2}dx\right) \quad ; \quad dx' = \gamma(dx - u\,dt) \quad ; \quad dy' = dy \quad ; \quad dz' = dz$$

Thence, one gets:
$$v'_{x'} = \frac{v_x - u}{1 - \frac{u}{V^2} v_x} \quad , \quad v'_{y'} = \frac{v_y}{\gamma\left(1 - \frac{u}{V^2} v_x\right)} \quad , \quad v'_{z'} = \frac{v_z}{\gamma\left(1 - \frac{u}{V^2} v_x\right)}$$



In the case where the P's motion occurs in such a way that z & z' are contant (just orient the spatial axes to get this situation!), it is easy to link the two speed modulous & directions of propagation defined w.r.t. observers at rest in F & F':

$$v_x = v\cos\theta \quad , \quad v_y = v\sin\theta \quad , \quad v'_{x'} = v'\cos\theta' \quad , \quad v'_{y'} = v'\sin\theta'$$

$$v'\cos\theta' = \frac{v\cos\theta - u}{1 - \frac{u}{V^2} v\cos\theta} \quad , \quad v'\sin\theta' = \frac{v\sin\theta}{\gamma\left(1 - \frac{u}{V^2} v\cos\theta\right)} \quad \rightarrow \quad \tan\theta' = \frac{v\sin\theta}{v\cos\theta - u}\sqrt{1 - \frac{u^2}{V^2}}$$

The case of a particle moving at Minkowski's speed (ie light in Maxwell's theory)

$$v_x^{\ 2} + v_y^{\ 2} = V^2 \quad \rightarrow \quad v'^{\ 2}_{x'} + v'^{\ 2}_{y'} = V^2 \quad \text{(check!)} \quad \rightarrow \quad \text{the vector has changed, but not its modulous}$$

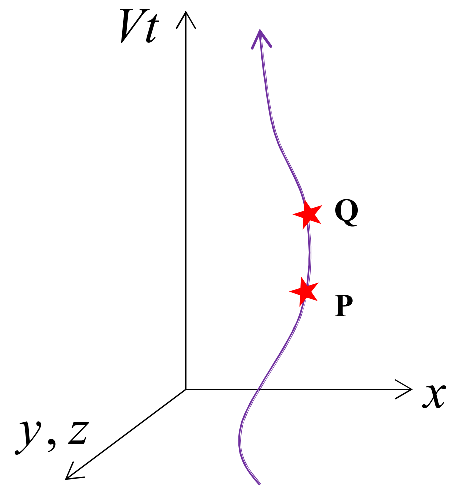
$$\cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta} \quad , \quad \sin\theta' = \frac{\sin\theta}{\gamma(1 - \beta\cos\theta)} \quad \rightarrow \quad \tan\theta' = \frac{\sin\theta}{\cos\theta - \beta}\sqrt{1 - \beta^2} \quad \text{with} \quad \beta = \frac{u}{V}$$

« aberration of light » formula

Spacetime diagrams

Aim: representing the different events (that may occur along a particle's orbit) in a diagram expliciting both time and space coordinates

Convention: display time t, or (better) *V*t, on the vertical axis, and « orthogonally » the space axes

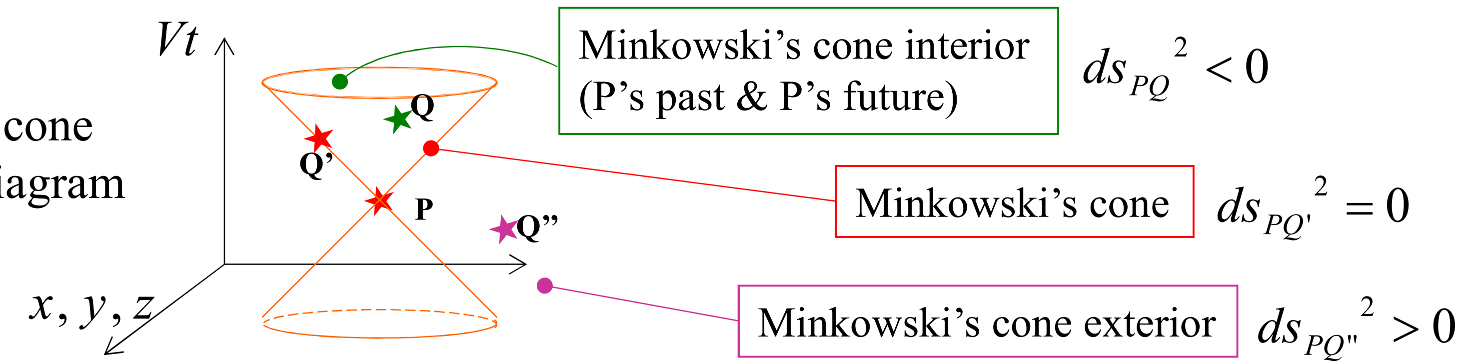


We will say that an orbit is **materializable** iff a **proper time** can be defined along it.
This requires that the galilean velocity has a modulous that is < V^2 at each of its point.

→ considering an event P, the other events belonging to a materializable orbit P belongs to are all in the interior of its Minkowski's cone, defined hereafter.

The **Minkowski's cone** attached to an event P is made of all the straight lines crossing it and along the which the Minkowski's interval is zero: $ds^2 = 0$.

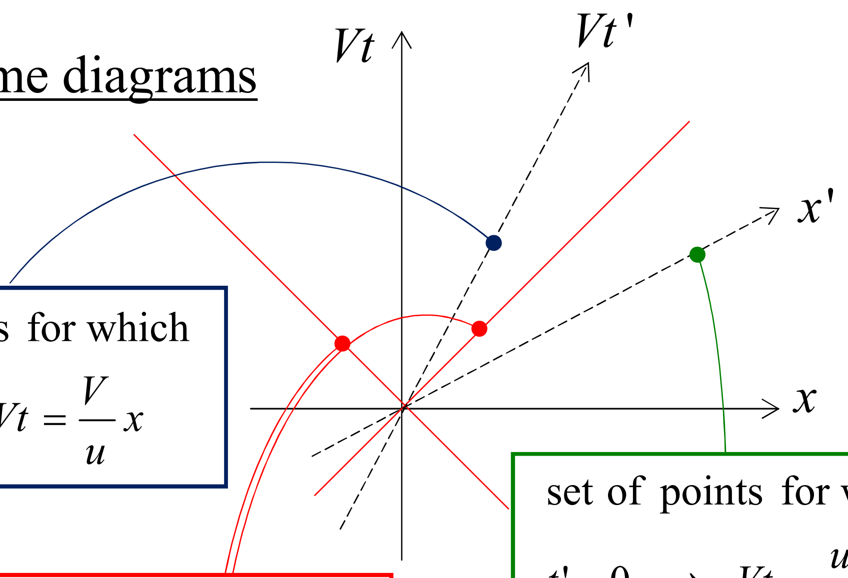
P's Minkowski's cone in a spacetime diagram



Lorentz special transforms versus spacetime diagrams

$$Vt' = \gamma\left(Vt - \frac{u}{V}x\right) \quad \text{with} \quad \gamma = \left(1 - \frac{u^2}{V^2}\right)^{-1/2}$$

$$x' = \gamma\left(x - \frac{u}{V}Vt\right)$$

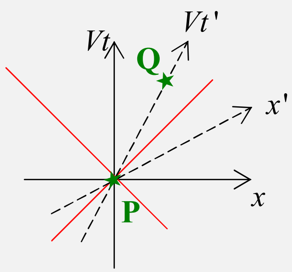


set of points for which $x' = 0 \rightarrow Vt = \frac{V}{u}x$

light cones imprints

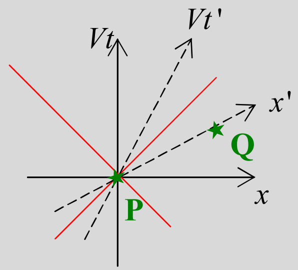
set of points for which $t' = 0 \rightarrow Vt = \frac{u}{V}x$

If $ds^2_{PQ} < 0$ there is a galilean frame in which PQ does not have space components → timelike vector



P & Q are time-ordered (in a way that does not depend on the observer)

If $ds^2_{PQ} > 0$ there is a galilean frame in which PQ does not have time components → spacelike vector



P & Q are NOT time-ordered … (depending on the observer, Q may occur after or before P)

If $ds^2_{PQ} = 0$ Q is on P's Minkowski's cone → null/isotropic vector

Causality

The **causality** **principle** claims that if an event P is the reason for an event Q, then **no « physical observer » can see the occurrence of Q before the one of P** (by a « physical observer », we mean an observer describing a materializable orbit, otherwise there would not be any proper time concept for this « observer »!)

→ P & Q can not define a spacelike vector

Thence, if the event Q is a physical consequence of the event P,
Q should occur inside or on P's Minkowski's cone.

Remark:
the fact that the causality principle could seem to be highly **natural** (in some sense) does **NOT** mean that it is logically **necessary** for a physical theory to be **coherent**!!!

Miscellaneous

1- « time dilatation », « length contraction », …: bad wordings!!!

→ at the origin of a lot of confusions, (pseudo-)paradoxes, … → *exercises!*

→ « length contraction »: in some sense, it is just another way to interpret time relativity

2- Simultaneity:

Time relativity → simultaneity relativity $Vt' = \gamma\left(Vt - \frac{u}{V}x\right)$ & $dt = 0 \rightarrow Vdt' = -\gamma\frac{u}{V}dx \neq 0$

→ claiming that 2 events occur simultaneously without any reference to a specified observer has just **no meaning** at all!

3- The GPS technology:

it would not even be imaginable in a non relativistic framework. Indeed, it requires:
- the existence of a speed the modulous of which does not depend on the transmitter/observer;
- the fact that the used medium (light) propagates at this modulous-invariant speed;
- keeping geolocalisation at the meter level requires taking explicitly time relativity effects into account. Indeed:

$v_{sat} \sim 4$ km/s → $(v/c)^2 \approx 2.10^{-10}$ → Desynchronisation rate ~ 15 µs/day → (multiply by c !) → Derive in positioning ~ 4 **km/day**!

(note that gravity induces effects of the same order)