Exercices

(The first number of the exercise's labelling refers to the lecture's number)

Ex 0-1 - Seemingly "faster than light" motions
A far object (star, galaxy, ...) throws matter (gaz) at a speed w, in the θ direction with respect to the line of sight, defined w.r.t. an observer.
1) Consider the photons emitted (at speed c!) by the thrown matter at times t and $t + dt$. What is the time interval δt between the corresponding receptions by the observer?
2) What is then the "projected on the sky velocity" measured by the observer (who just measures an angular velocity, but is supposed to know the distance of the source)?
3) Show that, depending on θ, this projected velocity can be $> c$ if w is $>$ to a minimum value $w_{\min}$, to be determined (and that is $< c$!).

Ex 1-1 - The rod and barn "paradox"
1) Consider a rod of length L_0 (in its rest frame). What seems to be the rod's length measured in a frame in which it is moving at the speed u (moving parallely to its length) ?
2) A runner crosses a barn of length L_0 (the barn having a door at each side) carrying a rod of length L_0 (parallely to the barn's length). When the runner is at the barn's center, the two doors are simultaneously, and instantaneously, closed and opened. What happens to the rod ? Analyse the situation from two points of views, and show that the conclusions are compatible (thence that there is no paradox ...).

Ex 1-2 - Accelerated frames & frozen time in special relativity
In a galilean frame, let O_{rest} be an observer at rest and O_{acc} another one experimenting a constant acceleration, as measured in the galilean frame having the same instantaneous velocity. (For questions 2 &3, a spacetime diagram may be helpful ...)
1) What is O_{acc}'s motion ?
2) Show that O_{rest} does not see O_{acc} at all times, but sees his whole life.
3) Show that O_{acc} always sees O_{rest}, but nevertheless sees only part of his life.

Ex 2-1 - Lagrange's equations
It may be useful to derive the Lagrange equations in the cases:
1) the lagrangian depends on two functions: $L\left[X\left(t\right), Y\left(t\right), X'\left(t\right), Y'\left(t\right)\right]$;
2) the dynamical function depends on two variables: $L\left[X\left(u,v\right), \partial_u X, \partial_v X\right]$ ...
3) ... and what about combining both?

Ex 2-2 - Gravitational Doppler
A transmitter P and a receiver Q are at rest in a Newton's metric of mass m. Let t_e and t_r be the emission and reception dates for a first photon, and $t_e + dt_e$ and $t_r + dt_r$ the dates for a second one.

1) Calculate the ratio dt_r/dt_e following the photon's path. (For convenience, one may consider that the mass m, the transmitter and the observer are aligned.)
2) Calculate the corresponding proper times ratio $d\tau_r/d\tau_e$.
3) In the case where r_P and $r_Q >> m$, expand the obtained expression at first order in m/r. From this expression, get a "neo-classical" interpretation of the Einstein's effect.

Ex 3-1 - Symmetry/antisymmetry
1) Show that any tensor T_{ij} can be written in the form $T_{(ij)} + T_{[ij]}$, where $T_{(ij)}$ is symmetric and $T_{[ij]}$ antisymmetric.
2) Let S^{ij} be a symmetric tensor and A^{ij} an antisymmetric one. Show that $S^{ij}T_{ij} = S^{ij}T_{(ij)}$ and $A^{ij}T_{ij} = A^{ij}T_{[ij]}$. Conclude that $S^{ij}A_{ij} = 0$.

Ex 3-2 - Useful formulae
Show that:
1) for any covariant vector $\nabla_i V_j - \nabla_j V_i = \partial_i V_j - \partial_j V_i$ (external derivative);
2) $\Gamma^i_{ij} = \partial_j \ln \sqrt{|g|}$ (admitting that $g^{ij}\partial_k g_{ij} = \frac{1}{g}\partial_k g$);
3) for any contravariant vector $\nabla_i A^i = \frac{1}{\sqrt{|g|}}\partial_i \left(\sqrt{|g|}A^i\right)$ (vectorial field divergence);
4) for any scalar $\nabla_i \nabla^i \Phi = \frac{1}{\sqrt{|g|}}\partial_i \left(\sqrt{|g|}g^{ik}\partial_k \Phi\right)$ (scalar field laplacian);
5) for any -1 weighted vectorial density $\nabla_i D^i = \partial_i D^i$.

Ex 3-3 - Ricci identity
Show that $\nabla_a g_{bc} = 0$.

Ex 3-4 - Geodesics in synchronous coordinates
Show that, in synchronous coordinates, the time curves (along which x^0 only varies) are (particular) geodesics.

Ex 3-5 - An other way to write the geodesics equation
Eliminating the λ affine parameter, show that the geodesics equation can be written as (with $t = x^0$)

$$\frac{d^2x^k}{dt^2} + \left(\Gamma^k_{\alpha\beta} - \frac{dx^k}{dt}\Gamma^0_{\alpha\beta}\right)\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0.$$

Ex 3-6 - The euclidean plane in polar coordinates & the sphere
Let us consider the 2 dimensional metric

$$ds^2 = du^2 + f\left(u\right)dv^2.$$

1) Calculate the connexion components.
2) Calculate the Riemann-Christoffel tensor components.

3) Calculate the Ricci tensor and the scalar curvature.
4) Calculate the Einstein tensor (remark : the result may sound surprising ... but it is a general result in 2 dimensions!)
5) Comment the three cases : $f(u) = 1$; $f(u) = u^2$; $f(u) = \sin^2 u$.
6) What happens if g_{ij} is replaced by $a^2 g_{ij}$ where a is a constant ?

Ex 4-1 - The Einstein's Universe and the cosmological constant
Using his gravity theory, Einstein tried to get a stationary and homogeneous cosmological solution, dust filled (dust = perfect pressureless fluid) and of finite volume. Thence, he looked for a metric

$$ds^2 = -dt^2 + \frac{dr^2}{1 - (r/R)^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

satisfying the RG equation for a dust fluid of density ϵ.
1) Show that in such a spacetime, time curves (r, θ, φ constant) are inertial orbits.
2) Show that the RG equations (without cosmological constant) do not admit such a solution if $\epsilon \neq 0$.
3) Show that the RG equations with a cosmological constant admit such solutions. Give the dependences between ϵ, R and Λ.
[hint : $\Gamma^1_{11} = r/\left(R^2 - r^2\right)$; $\Gamma^1_{22} = -r\left(1 - r^2/R^2\right)$; $\Gamma^2_{12} = 1/r$; $\Gamma^2_{33} = -\sin\theta\cos\theta$; ...]

Ex 4-2 - Einstein versus Poisson and local effect of Λ
Consider a weak field small spacetime region

$$g_{ab} = m_{ab} + h_{ab} \quad \text{with} \quad |h_{ab}| << 1.$$

The curvature tensors' components can be linearized in h_{ab}. Consider that the gravitational field is also quasi-stationary, which means that the time derivatives can be neglected w.r.t. the spatial ones.
1) Show that the Ricci's time-time component then reads

$$R_{00} \simeq \partial_k \Gamma^k_{00}.$$

2) Show that for low velocities, the spatial components of the geodesics equation approximate into

$$\frac{d^2 x^k}{dt^2} \simeq -\Gamma^k_{00}.$$

3) Show that it is legitimate to modelize slow velocity matter by a perfect fluid with $(u^a) \simeq (1, 0, 0, 0)$, and that its stress tensor components are $\simeq 0$ except for its time-time component, that reads $T_{00} \simeq \epsilon$.
4) Since, from the time-time Einstein's equation (with cosmological constant) component, one has $R_{00} = 8\pi G\left(T_{00} - \frac{1}{2} T g_{00}\right) + \Lambda g_{00}$ (check!), get

$$\partial_k\left(-\Gamma^k_{00}\right) = -4\pi G\epsilon + \Lambda$$

and conclude (1) that one recovers, to some extent, the Poisson equation, and (2) on the local effect of the cosmological constant.

Ex 4-3 - Perfect fluid's conservation laws
Consider the perfect fluid's stress tensor

$$T^{ab} = (\epsilon + p)\, u^a u^b + p g^{ab}$$

where ϵ, p and u^α are respectively the proper energy density, pressure and four-velocity fields of the fluid (remind that $u_a u^a = -1$).
1) Show that $u^a \nabla_b u_a = 0$.
2) Show that the stress tensor conservation yields

$$u^b \nabla_b \left[(\epsilon + p)\, u_a \right] + (\epsilon + p)\, u_a \nabla_b u^b + \partial_a p = 0$$

3) Show that if the fluid obeys a barotropic equation of state $p = p(\epsilon)$, one gets

$$\begin{aligned} \partial_\alpha \left(\sqrt{-g} r u^\alpha \right) &= 0 \\ (\epsilon + p)\, u^\alpha \nabla_\alpha u^\beta &= -\left(g^{\alpha\beta} + u^\alpha u^\beta \right) \partial_\alpha p \end{aligned}$$

where (so called rest proper energy density)

$$r = \exp \int \frac{d\epsilon}{\epsilon + p(\epsilon)}.$$

Ex 5-1 - Inertial orbit's energy in Schwarzschild
Show that, for any free motion in Schwarzschild, the quantity

$$E = \left(1 - \frac{2m}{r} \right) \frac{dt}{d\tau}$$

is conserved. Considering the weak field and slow velocity case, justify the "energy" labelling for this quantity.

Ex 5-2 - LCO and LSCO in Schwarzschild
(For all these questions, the (relativistic !) Binet's equation may be helpful ...)
1) Show that Schwarzschild admits circular inertial motions of radius r iff $r > r_{LCO}$ (LCO = Last Circular Orbit), and give r_{LCO}. By the way, why the labelling "photonic sphere" to the $r = r_{LCO}$ sphere ?
2) Show that Schwarzschild's circular orbits are stable iff $r > r_{LSCO}$ (LSCO = Last Stable Circular Orbit), and determine r_{LSCO}.
3) Show that for an inertial motion, r can not experiment a minimum inside the photonic sphere.